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ON
THE THEORY OF ALTERNATING CURRENTS

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A TREATISE
ON
THE THEORY OF ALTERNATING CURRENTS

by

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PREFACE.

THE great progress that has been made in the application of alternating currents to industrial purposes makes it desirable to collect together and examine the mathematical theorems which electricians use in their everyday work. In this volume the more general theorems are collected, and proofs are given of the more important of them, due stress being laid on the assumptions that it is sometimes necessary to make, and on the consequent limitations in the use of the formulae. In the second volume the theory of alternators, motors, transformers, converters, and of the transmission of power by polyphase currents will be given.

The reader is supposed to be familiar with the elementary theory of electricity and magnetism, and to have a working knowledge of trigonometry and of the elements of the calculus. A knowledge of De Moivre's theorem, De Moivre's property of the circle and of hyperbolic sines and cosines will be found essential. Lord Kelvin's bei and ber functions given in Chapter XVI will be understood at once from their definitions.

The references, given at the end of nearly every chapter, are to the books or papers I have consulted when writing the chapter. In many cases they contain more detailed discussions of the same or similar problems, and will be helpful to the student. The proofs of several of the problems could have been considerably shortened by deducing them from the general equations of electromagnetism, but, in my opinion, the proofs given bring out the physical meaning more clearly to the electrical student.

The early chapters deal mainly with inductance and capacity, and it is hoped that the practical formulae in them will be helpful to the working electrician. In Chapter V, illustrations are given of methods by means of which the capacities of polyphase cables and overhead wires can be calculated. It is also shown how the inductances of these combinations of conductors for the case of surface currents can be found. The definition and the theory of the power factor given in Chapter VI are now almost universally adopted by electricians. It was thought necessary to define terms like 'watt current' and 'wattless current'—'le courant watté' and 'le courant dewatté' of French writers—as they are so widely used. In Chapter IX, the test room methods of measuring power are given, and in Chapter X the method, when discussing practical problems, of replacing an air-core transformer by its equivalent net-work is explained. In Volume II this method will be extended to iron-core transformers.

In Chapter XII some problems in two phase theory are discussed graphically and illustrate how the theorems of solid geometry can be applied usefully in this case. In Chapter XIII the main problems in the theory of phase indicators and induction type watt-hour meters are stated, and approximate solutions are obtained. The theory of rotating magnetic fields, given in Chapter XIV, is founded on a paper which I wrote for the *Electrical Review* in 1893. In Chapter XV the interesting problem of the nature of the magnetic field round parallel wires, carrying polyphase currents, is discussed. Although complete solutions of the problem of the eddy currents in magnetic metals have not yet been obtained, the approximate solutions obtained by J. J. Thomson and Oliver Heaviside, which are given in Chapter XVI, will be found most helpful in practical work. Heaviside's solution is given in terms of bei and ber functions, and so, in given cases, numerical values can be found readily by means of the tables given in this volume. In Chapter XVII a slight sketch is given of the useful method of duality.

I have to thank Mr G. F. C. Searle, M.A., of St Peter's College, Cambridge, University Lecturer in Experimental Physics, for the many suggestions and emendations he has made, during the printing of the work. Of these suggestions and emendations I have freely availed myself. Particularly I have to acknowledge the valuable help he has given me in the problems connected with capacity, eddy currents and inductance, more especially the inductance when the currents are confined to the surface of the conductors. I have to thank him also for his unwearying kindness in reading the proofs and for checking nearly all the formulae given.

I take this opportunity of thanking Dr Charles Chree, F.R.S., the Superintendent of the Observatory Department of the National Physical Laboratory for his help and encouragement. I am also indebted to Mr W. C. Dampier Whetham, F.R.S., who has edited this work, for many valuable suggestions and criticisms.

Finally, I have to thank the Council of the Institution of Electrical Engineers and *The Electrician* Printing and Publishing Company for their kind permission to make what use I pleased, in the preparation of this volume, of any of my papers which they have published.

A. R.

2 BELLEVUE PLACE,
RICHMOND, SURREY.
October, 1904.

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ERRATA.

P. 9, line 21, after B insert ', B completely surrounding A, and.'

P. 51, line 14, for 'R' write '$\mathcal{R}$.'

P. 87, line 3, dele 'ultimately' and write ', since i_1 and q are zero at the instant of closing the switch in the main, A is zero, and thus.'

P. 87, line 21, dele 'ultimately' and add, 'since e_1 and $\int \frac{R}{L} e_2 dt$ are zero at the moment of closing the switch in the main, A is zero, and thus.'...

P. 192, line 12, dele 'therefore' and insert 'but, when the potentials are kept constant, the *gain* dW in the electrical energy (see p. 400) is equal to the work done on the disc, and is therefore given by.'

SYMBOLS.

A, effective value of an alternating current; a constant.
$A_1, A_2, \ldots$, effective values of the currents in polyphase mains.
B, magnetic induction.
$B_{max.}$, maximum value of the magnetic induction.
C, direct current; a constant.
E, maximum value of the alternating voltage; direct voltage.
$E_1, E_2, \ldots$, effective values of star voltages.
F, force.
G, average torque.
H, magnetic force.
I, intensity of magnetisation; maximum value of an alternating current when it follows the harmonic law.
$I_{max.}$, maximum value of an alternating current.
$I_1, I_2, \ldots$, effective values of the mesh currents.
K, capacity between two conductors; capacity of a conductor.
$K_{1.1}, K_{1.2}, \ldots$, Maxwell's coefficients of self and mutual induction for electrostatic charges.
K_q, capacity when the charges are constant.
K_v, capacity when the potentials are constant.
L, self inductance.
$L_{1.1}, L_{1.2}, \ldots$, coefficients of induction in electro-magnetism.
M, mutual inductance; magnetic moment.
P, power.
Q, quantity of electricity.
R, resultant electrostatic force; resistance; resistance of primary circuit; radial force.
R_1, resistance of primary circuit.
R_2, resistance of secondary circuit.
S, area of cross section; resistance of secondary circuit.
T, periodic time; tangential force; longitudinal tension.
V, electrostatic potential; magnetic potential; volume; effective value of potential difference.
$V_{1.2}, V_{1.3}, \ldots$, effective values of the mesh voltages.
W, average value of the power; power; energy.
Z, impedance.

c, thermal capacity per unit volume.
e, instantaneous value of the electromotive force.
$[e]$, $E \cos \omega t + \sqrt{-1} E \sin \omega t$.
f, force; frequency; fault resistance.
$f_1, f_2, \ldots$, fault resistances of the mains.

sketch is given of a method of duality founded on these analogies, and Maxwell has shown how important they are in advanced theory.

Electrostatics.

It is convenient to divide the electricities generated when certain bodies are rubbed together into two kinds, which are called positive and negative. It is found that a body charged with positive electricity will repel a body charged with the same kind of electricity and will attract a body negatively charged. By means of a torsion balance Coulomb proved that the force of repulsion f between two small bodies in air possessing electric charges q and q' of like sign is given by the formula

$$f = k\frac{qq'}{r^2},$$

where r is the distance between the bodies and k is a constant. If we define unit charge to be that charge which if concentrated at a point would repel an equal like charge concentrated at a point one centimetre away with a force of one dyne, then k is unity and the formula becomes

$$f = \frac{qq'}{r^2}.$$

If the bodies be not immersed in air we must write

$$f = \frac{1}{\lambda}\frac{qq'}{r^2},$$

where λ is a constant depending on the medium in which the bodies are placed. This constant is generally called the "specific inductive capacity" of the medium, but the term "dielectric coefficient" is coming into use.

Faraday mapped out the electric field which surrounds a charged body by means of lines drawn so that the direction of the resultant force at any point on these lines is in the direction of the tangent at that point. By the resultant force at a point is meant the force with which a unit positive charge placed at the point would be urged if it could be placed there without disturbing the electrical distribution elsewhere. These lines he called lines of force. We shall see later on that we can map out by means of tubes of force not only the direction of the field, but also its strength.

Potential.

The electric potential at a point P due to any electrified bodies in the neighbourhood is the amount of work in ergs that has to be done on a unit of positive electricity to bring it from the boundary of the field to P, the electric distribution being supposed to be undisturbed during the process. If the electric potentials at two neighbouring points P and P' be V and $V+dV$ respectively, and if PP' equal ds, then the work done by the electric forces while unit of positive electricity is moved from P to P' will be $V-(V+dV)$, and if F be the average electric force in dynes from P to P' the work done will also be represented by Fds.

Therefore $$Fds=-dV,$$

and $$F=-\frac{dV}{ds}.$$

Hence, if we know the mathematical expression for the potential at a point, this equation completely specifies the electric force in any direction at the point. We can therefore represent the attractions and repulsions by means of a single symbol V instead of having to give the components of the force at the point in three directions mutually at right angles. This is one of the advantages of the potential method of treating problems in attractions and repulsions. We should notice that the potential function itself is an undirected quantity; it possesses merely magnitude.

The electromotive force between two points P and Q is defined as the work done in taking a unit of positive electricity from one to the other. Thus if V_1 and V_2 be the potentials at P and Q, then

$$\text{the electromotive force between } P \text{ and } Q = V_1 - V_2$$
$$= \int_1^2 Fds.$$

It will be seen that the dimensions of electromotive force, or as it is generally written E.M.F., are the same as those of work divided by electrical quantity.

Suppose that we have q units of electricity concentrated at a point O and that we wish to find the potential V at a point P distant r from O.

By definition $$V=\int_r^\infty \frac{q}{\lambda r^2}dr=\frac{q}{\lambda r}.$$

If we have n charges q_1, q_2 ... at distances r_1, r_2 ... from O, then

$$V=\frac{q_1}{\lambda r_1}+\frac{q_2}{\lambda r_2}+\dots$$

$$=\Sigma\frac{q}{\lambda r}.$$

A surface on every point of which V is constant is called an equipotential surface. Since $\frac{dV}{ds}$ is obviously zero along such a surface, there will be no force tangential to it, and hence the lines of force must always cut it at right angles. Two equipotential surfaces cannot cut one another, as the same point cannot be at two different potentials.

Gauss's Theorem.

Let R (Fig. 1) be the intensity of the electric force, *i.e.* the force on unit quantity of electricity at a small element dS of a surface completely enclosing a charged body q. Let α be the angle which R makes with PN the outward normal to dS. Then $R\cos\alpha dS$ is called the flux of force across dS. If $d\omega$ be the solid angle which dS subtends at q, then

$$d\omega=\frac{dS\cos\alpha}{r^2}.$$

Now, if we take the sum of $R\cos\alpha dS$ over the whole surface, we get the total flux which traverses the surface. In symbols

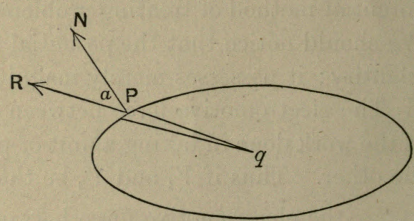

Fig. 1. Flux of force over the surface $=\Sigma R\cos\alpha dS=4\pi q$.

$$\Sigma R\cos\alpha dS=\Sigma\frac{q}{r^2}\cos\alpha dS$$

$$=\Sigma q d\omega$$

$$=4\pi q,$$

for q is constant and the sum of all the solid angles at a point is the ratio of the surface of a sphere to the square of its radius, *i.e.* 4π. If we had n particles with charges q_1, q_2 ... inside the closed surface, then we have by addition

$$\Sigma R\cos\alpha dS = \Sigma R_1\cos\alpha_1 dS + \Sigma R_2\cos\alpha_2 dS + \dots$$
$$= 4\pi(q_1 + q_2 + \dots + q_n).$$

This is true whatever the shape of the surface may be, provided that it embraces all the charges. It follows that, if we know $\Sigma R\cos\alpha dS$ over a closed surface, then we can find the sum of the charges enclosed by dividing this sum by 4π. It also follows that if there are no charges within the surface, then

$$\Sigma R\cos\alpha dS = 0.$$

Poisson's Equation.

If we choose three axes OX, OY and OZ at right angles to one another, and take the surface integral of the normal force over a small rectangular parallelopiped $dxdydz$, we get an important equation due to Poisson. Consider the part of the surface integral contributed by the two faces of the parallelopiped parallel to the plane YOZ. One side contributes $+\frac{dV}{dx}dydz$, and the other

$$-\frac{dV}{dx}dydz - \frac{d}{dx}\left\{\frac{dV}{dx}dydz\right\}dx.$$

Hence by addition we see that the surface integral over these two faces gives

$$-\frac{d^2V}{dx^2}dxdydz.$$

Proceeding similarly for the other four faces, and equating the total sum to $4\pi\rho dxdydz$, where ρ is the volume density of the distribution, we get Poisson's equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0.$$

This equation is generally written

$$\nabla^2 V + 4\pi\rho = 0,$$

or

$$\rho = -\frac{1}{4\pi}\nabla^2 V.$$

In the particular case when V is the potential at a point in free space, we get Laplace's equation

$$\nabla^2 V = 0.$$

If the electric field is uniform and the lines of force are all parallel to the axis of x, then this becomes

$$\frac{d^2 V}{dx^2} = 0.$$

$$\therefore \ V = A + Bx,$$

where A and B are constants.

If the field be symmetrical about an axis and if r be the distance from the axis of a point P in air at which the potential is V, then we can easily show that

$$\frac{d}{dr}\left(2\pi r \frac{dV}{dr}\right) = 0.$$

$$\therefore \ V = A + B \log r.$$

Tubes of force.

If we imagine a tube in which the end starts from a positively electrified body and the sides of which are formed by lines of force, we get what Faraday called a tube of induction. As we move along a line of force, the potential continually diminishes, and hence a line of force can never be a closed curve. We imagine then a tube of force as starting from a positively electrified surface, and ending at a negatively electrified surface. Consider the portion of a tube of force intercepted between two equipotential surfaces. Let dS and dS' be the intercepts on them, and let F and F' be the intensities of the force at the two surfaces respectively. By the intensity of the force at a point we mean the electric force that would be exerted on a unit quantity of electricity placed there. Applying Gauss's equation to this element, and noting that the sides of the tube add nothing to the integral $\Sigma R \cos \alpha dS$, since R is zero over the sides, we see that

$$FdS - F'dS' = 0.$$

Hence along a tube of force the product of the intensity of the force at a point and the area of the section of the equipotential surface through the point intercepted by the sides of the tube is constant. If the section of the positive surface from which the tube starts contains unit quantity of electricity, we get a unit tube. Therefore the force at a point can be measured by the number of unit tubes which pass through unit area of the equipotential surface at the point. Hence both the direction and the magnitude of the field can be mapped out by means of these unit tubes.

Coulomb's Law for the intensity of a force near a conductor.

Coulomb found experimentally that the resultant force near an electrified conductor was at right angles to its surface and that the magnitude of the force was proportional to the surface density. The exact relation between these two quantities can be found from the following considerations.

In order that the electricity on a conducting body may be in equilibrium, the E.M.F. between any two points must be zero; hence the potential must be constant throughout and equal to its surface value. Thus the bounding surface of the conductor is an equipotential surface, and the resultant force at points infinitely near it is normal to the surface and the force at all points in the conductor is zero. Consider now the surface integral of the normal force over a small closed surface formed by an element dS of an equipotential surface very close to the conductor, the tube of force through the boundary of dS, and a surface inside the conductor continuous with the tube and closing it. The tube of force is supposed to enclose a quantity $\sigma dS'$ of electricity spread over an area dS' on the surface of the conductor. We see that FdS is the value of the surface integral, for the sides of the tube and the surface in the conductor contribute nothing to it. Hence by Gauss's theorem

$$FdS = 4\pi\sigma dS'.$$

When the equipotential surface is infinitely close to the charged body $dS = dS'$; hence

$$F = 4\pi\sigma.$$

This numerical relation was proved by Poisson.

If V be the potential at a point near the surface, and dn be an element of the normal to dS drawn outwards, then

$$\sigma = \frac{F}{4\pi}$$

$$= -\frac{1}{4\pi}\frac{dV}{dn}.$$

If the body be immersed in a medium of which the dielectric coefficient is λ, then

$$\sigma = -\frac{\lambda}{4\pi}\frac{dV}{dn}.$$

Since FdS is constant along a tube of force, we see that the quantities of electricity at each end of the tube are equal in magnitude but opposite in sign. If a conductor be placed in an electric field, it clearly follows from the principles we have developed that those parts of it where tubes of induction stop must be negatively electrified and those parts where they begin must be positively electrified.

The potential V due to a body cannot have a maximum or a minimum value in free space, for if it had then $\Sigma F \cos \alpha dS$ taken round a small sphere enclosing the point would not be zero. Hence, if the potential be constant round a closed surface it will be constant at all points in that space, as otherwise it would have a maximum or a minimum value at some point in it.

If we have various charged bodies enclosed in a metallic envelope, the tubes of force starting from them will all terminate on the inside surface of the envelope, and hence the induced charge will be exactly equal and of opposite sign to the sum of the charges on the bodies.

Replacing a system of charged bodies by a conductor of which the boundary is an equipotential surface.

Green proved mathematically that if we suppose an equipotential surface replaced by a conductor having the same boundary as the surface and if it be electrified so that the surface density is given by

$$\sigma = -\frac{1}{4\pi}\frac{dV}{dn},$$

then the electricity on this conductor will be in equilibrium, and will produce the same potential at external points as the charges enclosed by the equipotential surface originally did.

A physical proof of this theorem can be given as follows. Suppose the equipotential surface replaced by a metal sheet coincident with it. If the metal be connected to earth, it will be at zero potential, and all points external to it will also be at zero potential. Let V be the potential at an external point due to the enclosed charges and V_1 be that due to the bound charge on the sheet, then $V+V_1=0$ and therefore $V_1=-V$. Also at a point on the conductor itself $v+v_1=0$ and $v_1=-v=$ constant by hypothesis; hence the electricity on the conductor would be in equilibrium. Therefore, if we suppose this conductor charged with electricity of the opposite sign to what it has in this case, it will produce a potential V at all external points and have itself a constant potential v. Hence the theorem follows.

Electric images. The method of images is due to Lord Kelvin. It is of great practical value in solving problems connected with the form of the lines of force round conductors suspended parallel to the earth, etc. Suppose for example we have a wire parallel to the earth at a distance h above it. Let the wire be charged with q units of electricity. We imagine an equal parallel wire at a depth $-h$ in the earth with a charge of $-q$ units. The surface of the earth will then be an equipotential surface of this system, and the distribution of the lines of force in the air can be found by solving the problem of two parallel wires at a distance $2h$ containing equal and opposite charges. Use will be made of this method in Chapter V.

Capacity. Consider two conductors A and B, both at an infinite distance from all other conductors. Suppose that a charge q be given to A and that B is always maintained at zero potential. When the electricity is in equilibrium on A, the surface density will be given at any point P by $\sigma=\frac{R}{4\pi}$ where R is the resultant force at the point P which is, by what we have seen, perpendicular to the conductor. All points on the conductor A will be at the same potential v, and we picture tubes of force all starting perpendicular to A and finishing up at right angles to B. The quantity of electricity on B will be $-q$. Now if we were to give to A a further charge q, then it would distribute itself over A so that the density would be now 2σ. The mutual actions between the two coincident charges will be perpendicular to the surface, and so doubling the density of the layer does not disturb the equilibrium. Hence also we shall have a charge

$-2q$ on B. The potential of A will now be $2v$, since σ, and therefore R, is doubled. Similarly if we gave a charge nq to A, its potential would become nv. We see then that the charge on A when B is maintained at zero potential, is in a constant ratio to its potential. This constant ratio is defined as the capacity of the conductor A. When there are several conductors at different potentials, the relations between them can be expressed by linear equations. We will consider this problem in Chapter IV.

Potential energy of an electrified body.

Consider the body A in the preceding paragraph. The relation between q and v may be written

$$q = Kv$$

where K is the capacity of A. Let dq be the increment of charge necessary to raise the potential to $v + dv$. The work done in taking dq from the boundary of the field to A is $dq \, . \, v$, therefore the work in ergs done in charging the conductor with a charge Q is

$$\int_0^Q vdq = \int_0^V Kvdv$$
$$= \tfrac{1}{2} KV^2$$
$$= \tfrac{1}{2} QV$$
$$= \tfrac{1}{2} \frac{Q^2}{K}.$$

Electric current.

When we have an E.M.F. between two points in a conductor, a current is produced. A current is measured by the rate at which quantity of electricity flows through any cross section of the conductor; it is therefore $\frac{dq}{dt}$ and will be denoted by i. The work done in taking q from a point where the potential is v_1 to a point where it is v_2 is

$$q(v_1 - v_2) = wt \text{ ergs},$$

where w is the work done per second in ergs, and t seconds is the time of working, the rate of working being uniform.

Hence differentiating

$$i(v_1 - v_2) = w.$$

In this equation i is the current in electrostatic units, $v_1 - v_2$ is the potential difference (P.D.) in electrostatic units and w is the power in ergs per second. The corresponding equation in electromagnetic units is

$$i(v_1 - v_2) = w,$$

where i is the current in amperes, $v_1 - v_2$ the P.D. in volts and w the power in joules per second or watts. The ampere, which we will define shortly, is 3×10^9 times the electrostatic unit of current and the volt is the three-hundredth part of the electrostatic unit of E.M.F. (Chapter IV.).

Magnetism.

If any magnet be supported in such a way that it is free to turn about its centre of gravity in the earth's magnetic field, it is found that a particular line through the centre of gravity of the magnet always tends to point in the same direction. This line is called the magnetic axis of the magnet. If we have a long thin cylindrical magnet with its magnetic axis coincident with the axis of the cylinder, then the centres of the circular faces are the poles of the magnet. To a first approximation we can suppose that these poles are centres of force, and the action of the magnet can be calculated by supposing attracting matter m concentrated at the north pole of the magnet, *i.e.* the pole which points to the north when the magnet is suspended, and repelling matter $-m$ concentrated at the south pole, the rest of the magnet consisting of inert matter. If $2l$ be the distance between the poles, $2lm$ is called the magnetic moment (M) of the magnet.

It is found by experiments with the torsion balance and otherwise that like magnetic poles repel one another and unlike attract with a force which is directly proportional to the strengths of the poles and inversely proportional to the square of the distance between them. We define the unit pole to be that pole which repels an equal like pole one centimetre away with a force of one dyne. Hence, in air or other non-magnetic medium, the law of repulsion of magnetic poles is

$$f = \frac{mm'}{r^2}.$$

Following the electrostatic analogy, we define the magnetic potential at a point to be the work done in ergs in taking unit north pole from the boundary of the field to the point in question. Hence if there was only one pole of strength m, the potential at a distance r from it would be given by

$$V = \frac{m}{r}.$$

Potential near a bar magnet.

Let N and S (Fig. 2) be the poles of a bar magnet. Let their strengths be m and $-m$ respectively and let $2a$ be the distance between them. Let O be the middle point of NS and let OP equal r. Then if V be the potential at P due to this magnet we have

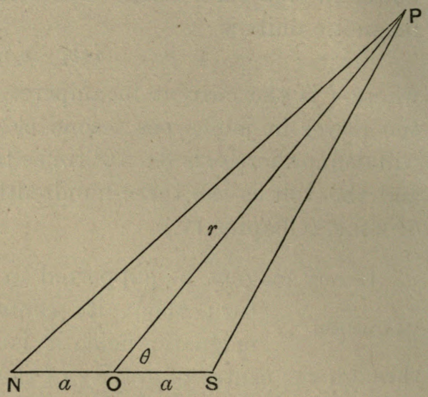

Fig. 2. Magnetic Potential at P
$= \frac{M\cos\theta}{r^2} + \frac{M(5\cos^3\theta - 3\cos\theta)\,a^2}{2r^4} + \ldots.$

$$V = \frac{m}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} - \frac{m}{\sqrt{a^2 + r^2 + 2ar\cos\theta}} \ldots\ldots (1).$$

If r is greater than $(\sqrt{2}+1)\,a$ we have by the binomial theorem,

$$\frac{1}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}$$

$$= \frac{1}{r}\left\{1 - \left(2\frac{a}{r}\cos\theta - \frac{a^2}{r^2}\right)\right\}^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left\{1 + \frac{1}{2}\left(2\frac{a}{r}\cos\theta - \frac{a^2}{r^2}\right) + \frac{3}{8}\left(2\frac{a}{r}\cos\theta - \frac{a^2}{r^2}\right)^2 + \ldots\right\}$$

$$= \frac{1}{r}\left\{1 + \cos\theta\,\frac{a}{r} + \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\frac{a^2}{r^2} + \left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\frac{a^3}{r^3} + \ldots\right\}.$$

Hence substituting in (1) and simplifying we get

$$V = \frac{2ma\cos\theta}{r^2} + \frac{5\cos^3\theta - 3\cos\theta}{r^4}\,ma^3 + \ldots$$

$$= \frac{M\cos\theta}{r^2} + \frac{M(5\cos^3\theta - 3\cos\theta)\,a^2}{2r^4} + \ldots,$$

where M is the magnetic moment of the magnet. Hence if r be large compared to a,

$$V = \frac{M \cos \theta}{r^2}.$$

It follows that the force in the direction OP will be

$$-\frac{dV}{dr} = \frac{2M \cos \theta}{r^3},$$

and in a direction at right angles to OP, in the direction in which θ increases, it will be

$$-\frac{dV}{rd\theta} = \frac{M \sin \theta}{r^3}.$$

Hence we can easily find the direction and the magnitude of the magnetic force at a point P when r is large compared to the length of the magnet.

The equipotential surfaces round an infinitely small magnet will be found from the equation

$$Vr^2 = M \cos \theta$$

or

$$V (x^2 + y^2)^{\frac{3}{2}} = Mx$$

by giving various values to V.

Tubes of force. Tubes of induction.

We can prove in exactly the same way as in electrostatics (Gauss's Theorem) that

$$\Sigma R \cos \alpha dS = 4\pi m$$

where the integral is taken over a closed surface enclosing poles the sum of whose strengths is m. R is the intensity of the magnetic force at dS and α is the angle between the direction of the force and the normal to the surface. Applying this theorem to the part of a tube of force intercepted between two equipotential surfaces, we find that $FdS = F'dS'$, where F is the intensity of the force at a point on dS. Hence we can suppose the magnetic field divided up into tubes of force in exactly the same way as we divided up the electrostatic field, and the tubes of force map out the intensity and direction of the magnetic force at any point.

The product $R \cos \alpha dS$ is called the flux of force through the area dS and we see from Gauss's theorem that the total flux of force from a unit magnetic pole is 4π. It is proposed to call the unit tube of force the maxwell.

Now when magnetic force acts on a medium, it produces in it magnetic induction. In air the number of tubes of induction produced by the magnetising force is the same as the number of tubes of force. In a magnetic medium the number of tubes of induction may be thousands of times greater than the number of tubes of force. The strength of the field at a point in the medium is the number of tubes of induction per square centimetre. One tube of induction per square centimetre is called a gauss, and this is the unit in which magnetic induction density is measured.

Intensity of Magnetisation. Permeability.

If we break into two portions a long thin bar magnet which has been uniformly magnetised and has poles of strength m and $-m$ respectively, it will be found that each portion is a magnet with poles of practically the same strength as the original magnet. The axes of the magnets are the two portions of the original axis. This is true also when we divide the magnet into many parts, and hence we can regard such a magnet as made up of a large number of little magnets of which the axes coincide with the axis of the original magnet and with ends perpendicular to that axis. Such a magnet is called a solenoidal magnet, and we see that no great error is made in assuming that its ends are covered with a layer of strength m of attracting and repelling matter respectively, and that the rest of the magnet is inert.

The intensity of the magnetisation of a solenoidal magnet is defined as the magnetic moment per unit volume, and is generally denoted by I.

Hence
$$I = \frac{M}{V} = \frac{ml}{Sl} = \frac{m}{S}$$

where S is the cross-sectional area of the magnet. We see that I may also be defined as the pole strength per square centimetre of the area of the cross section.

If the magnet were not uniformly magnetised, I would be different at different points, and so we should have to define it as

$$\frac{dM}{dV}.$$

Consider now a circular iron ring uniformly magnetised, the cross section of which is circular. If I be the intensity of the magnetisation, then, if this ring be sawn through we should have what may be regarded as a layer of attracting matter on one side of the air gap and a layer of repelling matter on the other, the surface density being in each case I. If a unit magnetic pole be placed in the air gap on the axis of the ring and at a distance a from either circular face, the force of attraction F to one face will be obviously along the axis, and therefore

$$F = \int_0^R \frac{2\pi r I dr}{a^2 + r^2} \times \cos\theta$$

where θ is the angle made by the axis with a line drawn from the pole to an element of the ring with a radius r. Now $r = a \tan\theta$,

therefore $$dr = a \sec^2\theta d\theta = \frac{1}{a}(a^2 + r^2)\, d\theta.$$

Therefore $$F = 2\pi I \int_0^\phi \sin\theta d\theta$$
$$= 2\pi I (1 - \cos\phi),$$

where ϕ is the value of θ when r is the radius of the cross section. Hence, when a is small, F is $2\pi I$, and since the repelling face will repel the pole with an equal force, we see that the intensity of the field in the air gap is $4\pi I$. Faraday showed that these tubes or, as they are more commonly called, lines of force are continuations of lines inside the iron which are called lines of magnetisation.

If we saw through the ring at some other point, and then imagined it stretched out straight, the ends of the first gap still remaining the same distance apart, we see that our unit magnetic pole will now be subjected to the attractions and repulsions of the poles at the other ends of the bar; hence the field in which it is situated will be weakened. If the bar were very long however we could neglect the demagnetising effects of the ends. If the bar were placed in a magnetising field of intensity H, the unit pole would be subjected to forces H and $4\pi I$, where I is the new intensity of the magnetisation. If B is the resultant of H and $4\pi I$, then B will be the strength of the field in the air gap. If H and $4\pi I$ are in the same direction, then

$$B = H + 4\pi I.$$

The ratio of I to H is called the susceptibility of the iron, and the ratio of B to H is called the permeability. It is this latter ratio that is generally wanted in practice; it is always denoted by μ, so that

$$B = \mu H.$$

When iron is magnetised to a certain induction density B, it is found that when the force is withdrawn a certain quantity B_0 is left remanent in the iron, and a certain coercive force has to be applied to get rid of B_0. We will return to this point when we discuss magnetic tests.

Magnetic shell. Mutual potential energy of two shells.

If we have a thin sheet of iron made up of an infinite number of little magnets with their axes perpendicular to the sheet and their like poles all pointing in the same direction, we get what is called a magnetic shell. A study of the properties of these shells is most helpful in understanding the theory of electrodynamics. The strength of a magnetic shell is its magnetic moment per unit area, so that, if g be its strength and h its thickness, $\frac{g}{h}$ is the polar strength of the face per unit area.

We will now find the mutual potential energy of two such shells, A and B. Consider an elementary small magnet of the shell A; its potential at a point P is $\frac{g_1 dS_1 \cos\theta}{r^2}$, *i.e.* $g_1 d\omega$ where ω is the solid angle at P. Integrating over the whole shell A, we see that $g_1\Omega_1$ is the potential at P due to the shell, where Ω_1 is the solid angle subtended by its boundary at P. If the strength of the shell B were g_2, then the polar strength of an element dS_2 at P would be $\frac{g_2}{dn} dS_2$, where dn is the thickness of the shell, and we have shown that the potential at P is $g_1\Omega_1$. Hence the work done in taking the polar element dS_2 from infinity to its position at P against the repulsion of the shell A would be

$$g_1 g_2 \frac{dS_2}{dn} \Omega_1,$$

and this expression gives the mutual potential energy of dS_2 and the shell A. Similarly the energy of the other pole of the elementary magnet the end of which is dS_2 will be

$$-g_1 g_2 \frac{dS_2}{dn}\left(\Omega_1 - \frac{d\Omega_1}{dn}\, dn\right).$$

Hence the total potential energy of the elementary magnet is

$$g_1 g_2 \frac{d\Omega_1}{dn}\, dS_2.$$

Therefore, integrating over the whole shell B, we find that the mutual potential energy is given by

$$g_2 \int \frac{dg_1\Omega_1}{dn}\, dS_2.$$

Now $-\dfrac{dg_1\Omega_1}{dn}$ is the resultant force measured normally at the shell B due to the shell A. Hence the integral gives the flux of force ϕ_1 through the shell B due to the shell A. Similarly if ϕ_2 were the flux of force through A caused by B, $-g_1\phi_2$ would be the mutual potential energy.

Therefore $$g_2\phi_1 = g_1\phi_2,$$

which is a remarkable reciprocal relation of great practical importance. When the strengths of the shells are equal, we see that the flux through A caused by B equals the flux through B caused by A.

Electrodynamics.

Oersted showed that when a wire carrying an electric current produced by a battery was brought near a magnetic needle, the needle was deflected. Hence an electric current produces a magnetic field.

We have seen that an infinitely small magnet, the magnetic moment of which is M, produces a potential $\dfrac{M\cos\theta}{r^2}$ at points at a distance r from it, where θ is the angle which r makes with the length of the bar. Weber proved experimentally that a small closed plane circuit carrying a current not only produced the same field but also was acted on by the same forces as a small solenoidal magnet with an axis perpendicular to the plane of the coil, provided that a certain relation held between the magnetic moment of the magnet, the area of the circuit, and the current flowing in it. If S be the area of the circuit, and i the strength of the current, then the potential at any point is $k\frac{Si\cos\theta}{r^2}$, where k is a constant. If the fields produced by the small magnet and the small circuit are the same, we must have $M=kSi$. Now in order to simplify our formulae as much as possible, we choose our unit of current so that k is unity, and therefore

$$M=Si.$$

Hence also
$$V=\frac{Si\cos\theta}{r^2}.$$

The potential V of the small circuit at a point P, distant r from its centre, may be written

$$V=i\omega$$

where ω is the small solid angle subtended by S at P. If we now suppose that an infinite number of these small circuits are all crowded together, forming a network, and that they are all carrying equal currents i flowing in the same sense round the meshes, then it is easy to see that

$$V=i\Sigma\omega=i\Omega$$

where Ω is the solid angle subtended by the boundary of all the small circuits. Where the elementary circuits touch one another we have equal currents flowing in opposite directions, and hence they are neutralised. We see then that this arrangement is equivalent to a circuit coinciding with the boundary of the small circuits and carrying a current i. It follows that the potential V at any point due to an electric circuit is always equal to $i\Omega$, where Ω is the solid angle which the circuit subtends at the point.

We have shown above that the potential due to a magnetic shell of strength g at a point P is given by

$$V=g\Omega$$

where Ω is the solid angle subtended by the boundary of the shell at the point. Hence Ampère's theorem follows, namely, that a magnetic shell and a closed circuit are equivalent if $g=i$.

Circular current.

Consider the case of a circle of wire of radius r carrying a current i. The potential V at a point P on its axis perpendicular to its plane will be $i\omega$. With centre P (Fig. 3) describe a sphere passing through the circle. The area of the spherical cap intercepted by this circle is $2\pi Rh$, where R is the radius of the sphere and h the height of the cap, hence

$$V = i\frac{2\pi Rh}{R^2}$$

$$= 2\pi i(1-\cos\theta)$$

$$= 2\pi i\left(1 - \frac{x}{\sqrt{x^2+r^2}}\right)$$

where x is the distance of the point P from the plane of the wire. Hence if F be the force on unit pole at P along the axis, we have

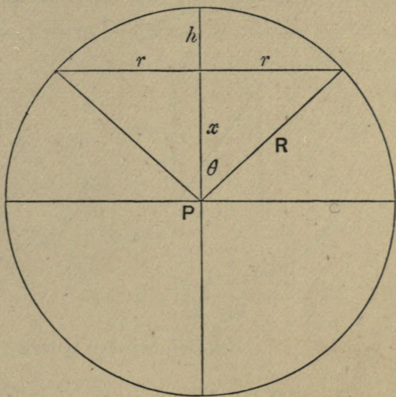

Fig. 3. Potential at $P = 2\pi i(1-\cos\theta)$.

$$F = -\frac{dV}{dx}$$

$$= \frac{2\pi i r^2}{(r^2+x^2)^{\frac{3}{2}}}.$$

When x is zero $$F = \frac{2\pi i}{r}.$$

In this formula F is in dynes, r in centimetres and i in C.G.S. units. We can thus define the unit current as the current which, flowing in a circle of radius r centimetres, produces a force of $\frac{2\pi}{r}$ dynes at its centre perpendicular to its plane.

The relation between the direction of the current and the lines of force can be remembered by the diagrams shown in Figs. 4 and 5. A current flowing in the direction of the arrowheads shown in Fig. 4 produces magnetic lines upwards through the paper.

In practice the unit current adopted is the ampere, which is one-tenth of the absolute C.G.S. unit defined above. A current of electricity in a conductor is the rate at which a quantity q of electricity is flowing through the conductor. In symbols

$$i = \frac{dq}{dt}.$$

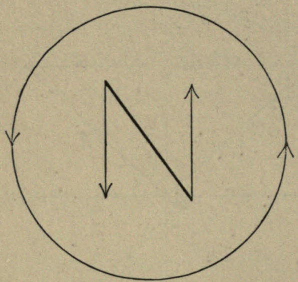

Fig. 4.

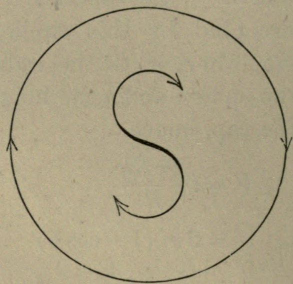

Fig. 5.

Relation between polarity and direction of current.

We may therefore define on the electromagnetic system the unit quantity of electricity as the quantity which flows past any cross section of the conductor every second when unit current is flowing. The practical unit of quantity is the coulomb, which is sometimes called the ampere-second.

Joule's Law.
Ohm's Law.

The quantity of electricity conveyed by a steady current of i amperes flowing for t seconds in a conductor between two points with a potential difference e, will be it. Hence the work done is

$$eit.$$

Now the practical electrical unit of work is the Joule or 10^7 ergs. The work done can be measured by the number H of units of heat (calories) generated in the conductor, and if the dynamical equivalent of heat be denoted by J (in joules per calorie), then

$$e = \frac{JH}{it} \quad \text{.............................(1).}$$

The unit in which e is measured is called the volt. If i were in centimetre-gram-second units and JH in ergs the number obtained for e would be 10^8 times larger. The volt equals 10^8 C.G.S. electromagnetic units. Equation (1) gives Joule's law in symbols.

A difference of potential between two points can be maintained by means of a battery; a current will flow between them, work being done in the process. Now Ohm proved experimentally that the ratio of the difference of potential to the current was constant so long as the conductor between the two points remained in the same physical state. This constant ratio is called the resistance of the conductor between the points, and Ohm's law may be written

$$i=\frac{e}{r} \qquad \text{.............................. (2).}$$

Hence if W be the number of joules done in time t,

$$W=JH=eit=\frac{e^2}{r}t=i^2rt.$$

If P be the rate at which work is being done between the two points, in joules per second, *i.e.* if P be the power in watts,

$$P=ei=\frac{e^2}{r}=i^2r.$$

Electromagnetic induction.

Whenever the lines (or tubes) of magnetic force passing through a circuit alter, an E.M.F. is set up in the circuit. We will first consider the case of a circuit carrying a current i in the neighbourhood of a magnet. Suppose that the flux of force through the circuit is ϕ, and that it alters to $\phi+d\phi$ in a time dt. Then the work done by the magnetic forces in moving the magnet is $id\phi$, and this amount of work must be supplied by the source of E.M.F. in the circuit. If E be the E.M.F. in the circuit, the work done in the time dt will be $Eidt$, and this work must be equal to the sum of $id\phi$ and the energy expended in heating the circuit, namely Ri^2dt.

Therefore $$Eidt=Ri^2dt+id\phi,$$

$$\therefore\ i=\frac{E-\dfrac{d\phi}{dt}}{R}.$$

Hence the induced E.M.F. in a circuit equals the rate at which the flux embraced by the circuit is altering; it acts in the positive direction (counter-clockwise) when the flux coming towards the spectator is diminishing.

This law, as modified by Maxwell, can be stated as follows.

Lenz's Law.

The E.M.F. generated in a circuit always tends to produce a current which opposes any change in the value of the flux. If we bring a positive magnetic pole near a circuit, we diminish the number of lines of force coming through it towards the spectator, and hence the induced current must flow in such a direction that the positive face of the equivalent magnetic shell may face the positive pole. Therefore the current must flow in the opposite direction to the hands of a watch (Fig. 4).

If the flux be increased, the induced current must tend to maintain the initial state of affairs, hence the direction of the induced E.M.F. is with the hands of a watch. To get the positive direction of rotation the flux must diminish, and hence the induced E.M.F. must be written $-\frac{d\phi}{dt}$. If ϕ be in maxwells, then $-\frac{d\phi}{dt}$ is in C.G.S. units. Hence if e be the counter E.M.F. in volts,

$$e = -\frac{d\phi}{dt}10^{-8}.$$

If ϕ traverse n turns of wire, then

$$e = -n\frac{d\phi}{dt}10^{-8}.$$

If we have two circuits A and B carrying currents i_1 and i_2

Mutual inductance.

respectively, then, since we may replace them by magnetic shells, we see that their mutual potential energy may be represented by $-i_1\phi_2$ or by $-i_2\phi_1$, where ϕ_2 is the flux through A due to B, and ϕ_1 is the flux through B due to A. Now ϕ_2 is in direct proportion to i_2; we may therefore write

$$\phi_2 = Mi_2,$$

where M is a constant depending only on the positions of the circuits. Similarly we can write $\phi_1 = M'i_1$, and, noting that the two expressions for the potential energy must be equal, we see that $M = M'$. Hence the flux of force through A due to unit current in B equals the flux of force through B due to unit current in A. If the current in B alters, the flux ϕ_2 through A alters also, and the induced E.M.F. set up is $-\frac{d\phi_2}{dt}$ or $-M\frac{di_2}{dt}$. It is to be noted that M may be positive or negative, depending on the directions taken as positive for the two currents.

In practice the circuits will consist of many turns of wire, and the fluxes through the turns will all be different. In this case the mutual potential energy of the two coils will be

$$-i_1\{\phi_2' + \phi_2'' + \dots\} = -i_1\Sigma\phi_2$$

where ϕ_2' is the flux due to i_2 embraced by one turn, ϕ_2'' the flux embraced by the next, and so on. The above definition applies only to non-magnetic circuits. The practical unit of mutual inductance is called the henry; it equals 10^9 times the C.G.S. unit.

Self Inductance.

If we double the strength of a current flowing in a non-magnetic circuit, we double the strength of the equivalent magnetic shell, and therefore we double the number of lines of force due to the current itself, embraced by the circuit. It follows that the current strength and the total flux are directly proportional to one another; we may thus write

$$\phi = Li$$

where L is a constant. This constant is called the self inductance of the circuit. We see that as i increases ϕ increases, and therefore by Lenz's law the induced E.M.F. tends to set up a current which will retard the increase of ϕ. The value of this E.M.F. is given by

$$e = -\frac{d\phi}{dt} = -L\frac{di}{dt}.$$

Instead of having a single loop, the circuit may consist of many loops; in this case we have

$$n_1\phi_1 + n_2\phi_2 + \dots = Li$$

where ϕ_1 is the flux embraced by a group of n_1 turns of wire, etc. Hence L may be defined as the number of linkages of the lines of force with the circuit when traversed by unit current.

The practical unit of self inductance like that of mutual inductance is the henry.

Electromagnetic energy.

Suppose that we have two circuits A and B respectively, and that L, N are their self inductances and M their mutual inductance. Then if i_1 and i_2 be the currents through the circuits, the flux through A is $Li_1 + Mi_2$, and similarly the flux through B is $Ni_2 + Mi_1$. If e_1 and e_2 be the E.M.F.'s in each circuit, and R_1, R_2 be their resistances,

$$e_1 = Ri_1 + \frac{d}{dt}(Li_1 + Mi_2)$$

$$= Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt}.$$

Similarly

$$e_2 = Ri_2 + N\frac{di_2}{dt} + M\frac{di_1}{dt}.$$

Thus,

$$e_1 i_1 + e_2 i_2 = R_1 i_1^2 + R_2 i_2^2 + \frac{d}{dt}(\tfrac{1}{2}Li_1^2 + Mi_1 i_2 + \tfrac{1}{2}Ni_2^2).$$

Now $R_1 i_1^2 + R_2 i_2^2$ is the rate at which energy is being expended in heat, and hence we see that $\frac{d}{dt}(\frac{1}{2}Li_1^2 + Mi_1 i_2 + \frac{1}{2}Ni_2^2)$ must be the rate at which energy is being stored up in the field. Hence when the currents in the two coils are i_1 and i_2, the energy stored up in the field is

$$\tfrac{1}{2}Li_1^2 + Mi_1 i_2 + \tfrac{1}{2}Ni_2^2.$$

This expression may be written in the form

$$\tfrac{1}{2}L\left(i_1 + \frac{M}{L}i_2\right)^2 + \tfrac{1}{2}i_2^2\frac{NL - M^2}{L};$$

and since it must be positive for all values of i_1 and i_2, and therefore when $i_1 = -\frac{M}{L}i_2$, we see that M^2 cannot be greater than LN.

When there is only one circuit, we see that the energy stored up in the field is $\frac{1}{2}Li^2$ when the current is i. By the last section we see that this may be written $\frac{1}{2}i\Sigma n\phi$ where ϕ_1 is the flux embraced by n_1 turns, etc.

The self energy of an electric circuit.

The above formulae and definitions only apply strictly to elementary tubes or filaments of current. The electric currents that we have to consider in practice cannot be regarded merely as filaments of current. In the general case we may write $\frac{1}{2}LI^2$ for the self energy stored up in the field, where I is the total current. A useful expression for this energy can be found as follows. Consider first a circuit made up of two filaments of current

i_1 and i_2, so that I equals $i_1 + i_2$. The self energy of the circuit is

$$\tfrac{1}{2} i_1 \phi_{1.1} + i_1 \phi_{2.1} + \tfrac{1}{2} i_2 \phi_{2.2},$$

where $\phi_{1.1}$ and $\phi_{2.2}$ are the fluxes due to the currents i_1 and i_2 respectively, and $\phi_{2.1}$ is the flux due to the current i_2 which is linked with i_1. Noting that $i_1 \phi_{2.1} = i_2 \phi_{1.2}$, we have

$$\tfrac{1}{2} L (i_1 + i_2)^2 = \tfrac{1}{2} i_1 (\phi_{1.1} + \phi_{2.1}) + \tfrac{1}{2} i_2 (\phi_{2.2} + \phi_{1.2}) = \tfrac{1}{2} i_1 \Sigma\phi_1 + \tfrac{1}{2} i_2 \Sigma\phi_2,$$

where $\Sigma\phi_1$ and $\Sigma\phi_2$ are the total fluxes linked with i_1 and i_2 respectively. When we have n filaments of current, we get in the same way

$$\tfrac{1}{2} LI^2 = \tfrac{1}{2} i_1 \Sigma\phi_1 + \tfrac{1}{2} i_2 \Sigma\phi_2 + \ldots = \tfrac{1}{2} \Sigma\Sigma\phi i;$$

the summations being effected over the whole system for all the filaments of current and the fluxes they embrace.

Self energy in terms of magnetic force. Kelvin's Formula.

The magnetic potential at a point due to a closed tube of current i is $i\omega$, where ω is the solid angle subtended at the point by the tube. If we take a unit positive pole once round a line of force embracing the tube, the change of potential energy is $4\pi i$, since 4π is the difference between the initial and final values of the solid angle. Hence if H be the magnetic force at any point

$$4\pi i = \int H ds,$$

where ds is an element of the line of force round which the integral has been taken.

Now if $d\phi$ be the flux of force over an element of an equipotential surface through the point

$$d\phi = HdS.$$

Hence $$id\phi = \frac{1}{4\pi} \int Hds \times HdS.$$

But HdS is constant along a tube of force, and $ds \times dS = dv =$ an element of volume of the tube of force.

Thus $$id\phi = \frac{1}{4\pi} \int H^2 dv,$$

the integration being taken along a tube of force.

Hence, integrating over all the equipotential surface, we get

$$i\phi = \frac{1}{4\pi} \Sigma H^2 dv,$$

where the summation is taken throughout all space.

Hence the expression $\frac{1}{2} i\phi$ for the self energy of the system may be written

$$\Sigma \frac{H^2}{8\pi} dv.$$

The E.M.F. generated in a conductor cutting lines of force.

Suppose that we have a wire sliding with uniform velocity v on two parallel wires which are joined by another wire at one end, and that the moving wire is cutting a uniform magnetic field which makes an angle θ with the plane of the wires. If the strength of the field be H, and the slider be perpendicular to the parallel wires, then the number of lines of force cut by it in t seconds is

$$H \sin \theta lvt,$$

where l is the length of the slider the ends of which are on the parallel wires. The increase of the flux round the closed circuit is $H \sin \theta lvt$. Therefore the E.M.F. generated will be

$$e = -\frac{d}{dt}(H \sin \theta lvt) = -H \sin \theta lv.$$

The E.M.F. generated is in the clockwise direction round the circuit. In this formula e is in C.G.S. units, H is in gausses, l in centimetres and v in centimetres per second. If e is in volts, then

$$e = -H \sin \theta lv \times 10^{-8}.$$

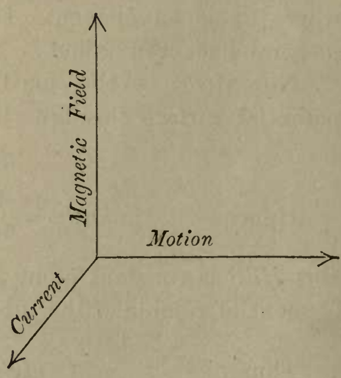

Fig. 6. Relation between the motion of a wire and the induced current.

In Fig. 6 is shown the relation between the motion of a wire, the magnetic field, and the direction of the induced E.M.F. and current. If we place the fingers of the right hand so that the thumb is in the direction of the motion and the first finger in the direction of the magnetic field, then, if the second finger be put at right angles to the first and the thumb placed so as to coincide with the wire, the second finger will point out the direction of the induced E.M.F. This E.M.F. will vanish when the motion is parallel to the field of force.

Force on a moving wire.

If the current in the moving wire be i, then the electric power generated by it will be ei. Hence if f be the resultant force in dynes acting on the wire, and v be expressed in centimetres per second,

$$fv = ei$$
$$= -H \sin \theta l v i,$$
$$\therefore f = -H \sin \theta l i,$$

where f is in dynes and i is also in absolute units. If i be in amperes, then

$$f = -\frac{H \sin \theta l i}{10} \text{ dynes.}$$

The negative sign shows that there is a force acting on the wire tending to make it move so as to increase the number of lines of force embraced by the circuit. This is the principle that is utilised in the electric motor. We see then that a wire carrying a current and placed in a magnetic field is subjected to a certain force acting in a certain direction. If we move the wire in the direction opposite to that in which this force is acting, by Lenz's law the induced E.M.F. must act so as to increase the current flowing and thus impede the motion. Therefore we see that the force would be in the opposite direction to the line marked *motion* in Fig. 6. Fleming's left-hand rule states this in an easily remembered form. If the fore-finger of the left hand point in the direction of the field, the second finger in the direction of the current in the wire, then the thumb will point in the direction of the force on the wire, if the two fingers be held at right angles to one another and also to the thumb.

The magnetic force due to an element of current. Laplace's Formula.

It will be sufficient for our next purpose to suppose that the current is in a plane. We will calculate the magnetic force at a point O (Fig. 7) due to a current i in a closed circuit. Suppose that we have a positive pole of strength m at O and that

OA is a fixed line. The strength of the magnetic field at P due to O will be $\frac{m}{r^2}$. Hence if the arrow indicate the direction of the

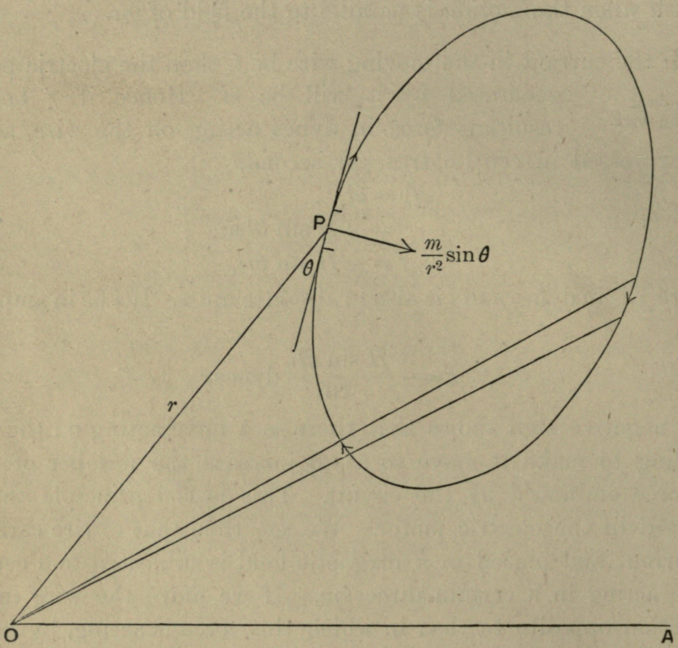

Fig. 7. Magnetic force at O due to the current i in a plane closed curve is upwards from the plane of the paper and

$$=\int\frac{i\sin\theta ds}{r^2}=\int\frac{id\phi}{r},$$

where ϕ = the angle POA.

current we see by the preceding paragraph that the force dF on ds will be given by

$$dF=\frac{m}{r^2}\sin\theta i ds \text{ dynes},$$

and the force, by the left-hand rule, tends to move ds downwards at right angles to the paper. Using a well-known artifice in Statics, we may replace this force by a force dF at O acting downwards, and a couple with a moment rdF round an axis lying in the plane of the paper and perpendicular to OP. It is easy to see that

$$rdF = \frac{m}{r} \sin \theta ids$$

$$= \frac{m}{r} \frac{rd\phi}{ds} ids$$

$$= mid\phi,$$

and hence $$\Sigma rdF = \int mid\phi,$$

for $\sin \theta = \frac{rd\phi}{ds}$, where the angle $POA = \phi$.

But round a closed curve for every couple of which the moment is $mid\phi$ there is another having a moment $-mid\phi$ (see Fig. 7); hence the resultant couple is zero. Thus the resultant force of the pole on the circuit is a force

$$\int \frac{m \sin \theta ids}{r^2}$$

acting downwards at O. Since action and reaction are equal and opposite, we see that the resultant force of the circuit on the pole is

$$\int \frac{mi \sin \theta ds}{r^2}$$

and acts upwards. We see then that, so long as we consider closed curves, we can calculate the intensity of the magnetic force F at any point by the formula

$$F = \int \frac{i \sin \theta ds}{r^2}$$

$$= \int \frac{id\phi}{r}.$$

If i be in amperes, then the magnetic force is given by

$$F = \frac{1}{10} \int \frac{id\phi}{r}.$$

Formulae for the magnetic forces inside circles and rectangles.

The force at the centre of a circle of radius R can easily be found, for $r = R$ and is constant, and $\Sigma d\phi = 2\pi$, hence

$$F = \frac{2\pi i}{R},$$

which agrees with our former result on page 19.

If the point O be not at the centre of the circle (Fig. 8) let $OC=a$, $CP=R$ and the angle $POC=\theta$, then

$$r = OP = a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}.$$

Therefore $$\frac{1}{r} = \frac{-a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}}{R^2 - a^2}.$$

But $$F = i\int_0^{2\pi}\frac{d\theta}{r}.$$

Thus $$F = \frac{i}{R^2 - a^2}\int_0^{2\pi}\sqrt{R^2 - a^2\sin^2\theta}\,d\theta$$

$$= \frac{i \times \text{circumference of ellipse}}{R^2 - a^2} \quad\text{...............(3).}$$

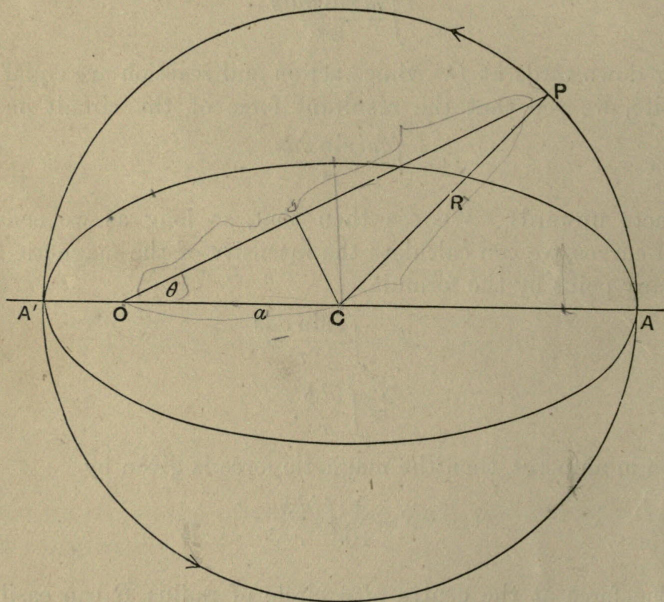

Fig. 8. The magnetic force at O due to a current i in the circle is perpendicular to the plane of the paper and equals

$$\frac{i \times \text{circumference of ellipse}}{R^2 - a^2}.$$

The ellipse has O for a focus. The force acts towards the reader.

The ellipse referred to in the formula has C for its centre, O for a focus, and has its major axis equal to a diameter of the circle. When a is zero this reduces to formula (2).

The case of a rectangular wire (Fig. 9) also admits of an easy solution.

If the current i be flowing round the wire in the direction indicated by the arrowheads, then the force F on unit pole placed at O will be given by the formula

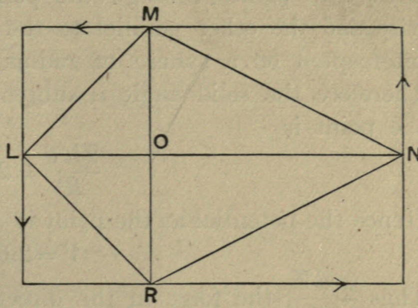

Fig. 9. Force at O due to current in rectangle $= i \times$ sum of reciprocals of perpendiculars from O on LM, MN, NR and RL.

$$F = i\left\{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right\} \quad \text{..................(4)},$$

where p_1, p_2, p_3 and p_4 are the perpendiculars from O on the lines LM, MN, NR and RL respectively, and ROM and NOL are drawn parallel to the sides of the rectangle. It is an instructive exercise to find graphically by means of this construction the density of the lines of force at various points inside a rectangle.

Force near a long straight wire carrying a current.

Let three of the sides of the rectangle move to infinity, then the formula becomes

$$F = \frac{2i}{r}$$

and we obtain the force due to a long straight current, where r is the perpendicular distance from the point considered to the wire. The force F is perpendicular to r, and the directions of force and current are related in the same way as the directions of current and force in Figs. 4 and 5. If the current be upwards through the plane of the paper, then the lines of force are circles, and act in the direction *N*. If the current be downwards through the paper, then the lines of force act in the direction *S*.

By Ampère's theorem, we could have replaced the wire by a plane magnetic shell bounded by it and extending to infinity.

Suppose that r makes an angle θ with the plane of this shell. Draw two planes through the point, one passing through the wire and the other parallel to the shell. The area of the lune intercepted on a sphere of radius R by these planes is $2R^2\theta$. Therefore the solid angle ω subtended by the infinite plane at the point is

$$\frac{2R^2\theta}{R^2} = 2\theta.$$

Hence the potential at the point is

$$V = 2i\theta.$$

Thus $-\frac{dV}{dr}$, the force in the direction of r, must be zero, and $-\frac{dV}{rd\theta}$, the force perpendicular to r in the direction in which θ increases, must be $-\frac{2i}{r}$. We see again that the work done in taking unit pole round the wire is

$$2i(2\pi + \theta) - 2i\theta = 4\pi i.$$

The magnetic force outside an infinite cylindrical tube carrying a current parallel to its axis.

Let PP' (Fig. 10) be a section of the cylindrical shell, and O the point at which we wish to find the force. Divide the shell into an infinite number of filaments parallel to its axis. Consider the force at O due to the currents in the filaments passing into the paper at P and P' respectively. From symmetry, their resultant will be perpendicular to OC, and hence, since one half of the shell is an image of the other, we see that the total force F acts per-

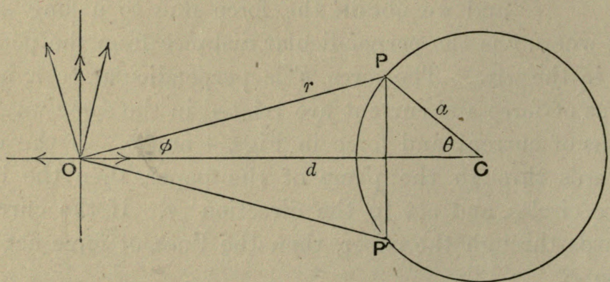

Fig. 10. The force at points outside a cylindrical sheet of current is the same as if the current were concentrated along the axis of the cylinder.

pendicularly to CO. Hence if i be the total current in the conductor, the angles PCO, POC be θ, ϕ respectively, and CP, PO and OC be a, r and d, then

$$F = 2\int_0^\pi \frac{2\frac{i}{2\pi}d\theta}{r}\cos\phi$$

$$= \frac{2i}{\pi}\int_0^\pi \frac{r\cos\phi}{r^2}d\theta$$

$$= \frac{2i}{\pi}\int_0^\pi \frac{d - a\cos\theta}{d^2 - 2ad\cos\theta + a^2}d\theta.$$

Now by expanding in a trigonometrical series (page 58) we see that

$$\int_0^\pi \log(d^2 - 2ad\cos\theta + a^2)\,d\theta = 2\pi\log d,$$

since d is greater than a.

Hence, differentiating with regard to d,

$$\int_0^\pi \frac{d - a\cos\theta}{d^2 - 2ad\cos\theta + a^2}d\theta = \frac{\pi}{d};$$

therefore $$F = \frac{2i}{d}.$$

Hence the cylindrical conductor acts as if its current were concentrated along its axis. As we can suppose that a solid cylindrical conductor is made up of an infinite number of coaxial tubes, we see that a solid cylinder also acts on external points as if its current were concentrated at the axis.

The magnetic force inside an infinite cylindrical tube carrying a current parallel to its axis.

Inside an infinite cylindrical tube carrying a current parallel to the axis, since d is less than a,

$$\int_0^\pi \log(d^2 - 2ad\cos\theta + a^2)\,d\theta = 2\pi\log a,$$

and hence, when we differentiate with regard to d, the right-hand side vanishes, and we get

$$F = 0.$$

Therefore the magnetic potential inside a hollow cylindrical conductor is constant, while outside it is given by

$$V = 2i\theta + A$$

where A is a constant.

Magnetic tests of iron. Hysteresis.

Let us wind an iron ring uniformly with insulated wire, and place a further winding on it connected to the terminals of a ballistic galvanometer. If we start a current through the first winding, the throw of the ballistic galvanometer enables us to calculate the flux of induction caused by the magnetising force of the current. If ϕ be the total flux in the core the induced current i in amperes will be given by

$$i = \frac{-n\frac{d\phi}{dt}}{R} \times 10^{-8},$$

if there are n turns of wire in series with the galvanometer, and R is the total resistance of this circuit in ohms.

Thus we have $$\int i dt = -\frac{n 10^{-8}}{R} \int d\phi,$$

and $$Q = \frac{n 10^{-8}}{R} (\phi_1 - \phi_2),$$

where Q is the number of coulombs that traverse the circuit when the flux increases from ϕ_1 to ϕ_2. The throw of the galvanometer needle gives the value of Q.

If we divide the flux by the cross section in square centimetres, we get the mean flux density B, and we shall prove in the next chapter that we can calculate the mean value of H by the formula

$$H = \frac{4\pi n_1 i}{10 l},$$

where n_1 is the number of turns in the primary winding which carries a current of i amperes, and l is the mean circumference of the iron ring.

If the magnetisation of the iron ring be reversed several times, until it gets into what is called its steady cyclic state, we get curves like those shown in Fig. 11.

Suppose that, when the current has its maximum positive value, OA is the value of H, and QA is the corresponding value of B. Then, as the current diminishes, the extremity of the ordinate representing B moves along QR. When the current is zero, the density of the flux left in the ring is represented by OR, and is called the remanence. When the flux is zero, H is OS, and this quantity is called the coercive force.

An inspection of the figure will show that, when the current diminishes from its highest value to zero, the change in the flux density is only $AQ - OR$, but in increasing numerically from zero

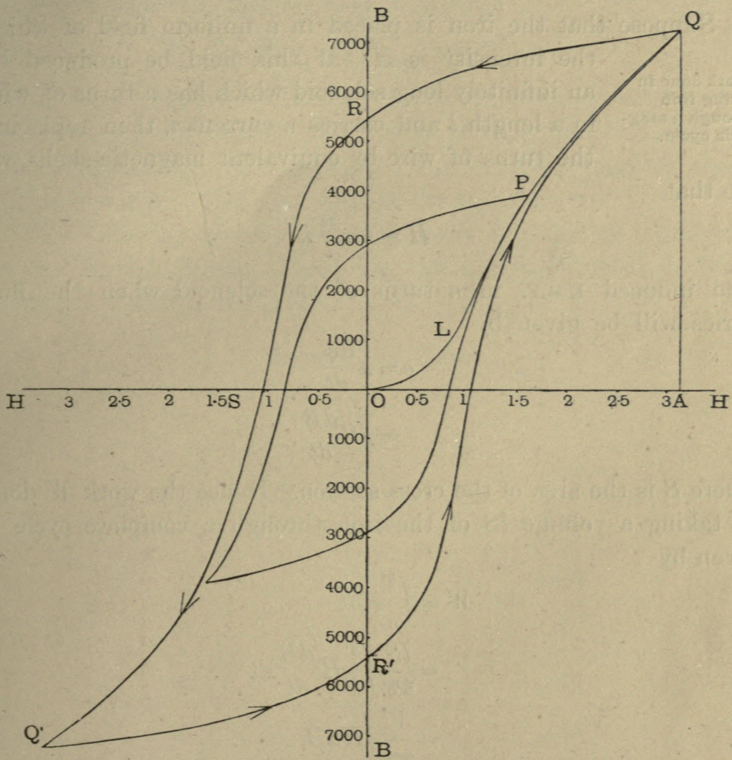

Fig. 11. Curves of Magnetisation of steel strips. $OLPQ$ = Permeability Curve. Hysteresis loops with $B_{\text{max.}} = 3900$ and $B_{\text{max.}} = 7300$.

to its maximum negative value it changes from OR to $-AQ$ or by an amount $AQ + OR$. On diminishing the current again, the extremity of the ordinate representing B moves along $Q'R'$. Finally on increasing the current up to its maximum positive value, the point arrives at the point Q from which it originally started. It will be seen that the magnetic induction lags behind the magnetic force producing it. This phenomenon is called hysteresis.

The curve OPQ in the figure gives the positions of the top points of the cyclic curves corresponding to various maximum values of H. The permeability μ of the iron for various values of H is found from this curve.

Work done in taking iron through a magnetic cycle.

Suppose that the iron is placed in a uniform field of which the intensity is H. If this field be produced by an infinitely long solenoid which has n turns of wire in a length l and carries a current i, then, replacing the turns of wire by equivalent magnetic shells, we see that

$$H = 4\pi \frac{n}{l} i.$$

The induced E.M.F. in n turns of the solenoid when the flux varies will be given by

$$e = n\frac{d\phi}{dt}$$
$$= nS\frac{dB}{dt},$$

where S is the area of the cross section. Hence the work W done in taking a volume lS of the iron through a complete cycle is given by

$$W = \int_0^T eidt$$
$$= \frac{lS}{4\pi}\int_0^T H\frac{dB}{dt}dt$$
$$= \frac{V}{4\pi}\int HdB,$$

where V is the volume of the iron in cubic centimetres, and the integral is taken over a whole cycle. From Fig. 11 we see that $\int HdB$ gives the area of the hysteresis loop corresponding to a given maximum value of B. If B be measured in C.G.S. units, W will be in ergs.

Steinmetz's Formula.

The following empirical formula due to Steinmetz is found of great use in practice,

$$W = \eta V B_{max.}^{1 \cdot 6},$$

where W represents the ergs lost in taking V cubic centimetres of iron from a maximum induction density of $B_{max.}$ to a density

$-B_{max.}$, and then back again to $B_{max.}$. The constant η depends on the kind of iron used.

If f be the frequency of the alternating current which magnetises the iron, the formula may be written

$$W = \eta f V B^{1\cdot6}_{max.} \times 10^{-7},$$

where W now represents the power lost in joules per second, or in watts owing to hysteresis. For values of $B_{max.}$ between 2000 and 14000, this formula is sufficiently trustworthy for practical work. For good soft iron and steel η is usually less than 0·002.

The following figures give the result of a test on steel strips for the British Electric Plant Co., the curves obtained for these strips being shown in Fig. 11.

Permeability Table.

H	B	μ
1	1580	1580
2	4930	2465
3	7000	2333
10	11,800	1180
20	13,840	692
50	15,800	316
100	16,600	166
160	17,980	112

Hysteresis.

Maximum B	Ergs per c.c. per cycle	Watts per c.c. at 100 ~ per sec.	Remanence	Coercive force	Steinmetz's coefficient η
3900	864	0·00864	2920	0·82	0·00152
7300	2386	0·02386	5470	1·08	0·00157

Taking $\eta = 0{\cdot}0015$ we see that the loss in watts per kilogramme of iron of which the specific gravity is 7·8, when $B_{max.} = 4000$ and $f = 100$, would be

$$= \eta f V B_{max.}^{1{\cdot}6} \times 10^{-7}$$

$$= 0{\cdot}0015 \times 100 \times \frac{1000}{7{\cdot}8} \times (4000)^{1{\cdot}6} \times 10^{-7}$$

$$= 1{\cdot}1 \text{ nearly.}$$

Number	1·5th power	1·55th power	1·6th power
1000	31,620	44,670	63,100
2000	89,440	130,800	191,300
3000	164,300	245,200	365,900
4000	253,000	383,000	579,800
5000	353,600	541,300	828,600
6000	464,800	718,000	1,109,000
7000	585,700	911,800	1,420,000
8000	715,600	1,122,000	1,758,000
9000	853,800	1,346,000	2,122,000
10,000	1,000,000	1,585,000	2,512,000
11,000	1,154,000	1,837,000	2,926,000
12,000	1,315,000	2,103,000	3,363,000
13,000	1,482,000	2,380,000	3,822,000
14,000	1,657,000	2,670,000	4,303,000
15,000	1,837,000	2,971,000	4,806,000
16,000	2,024,000	3,284,000	5,328,000
17,000	2,217,000	3,607,000	5,871,000
18,000	2,415,000	3,942,000	6,433,000
19,000	2,619,000	4,286,000	7,015,000
20,000	2,828,000	4,641,000	7,615,000
21,000	3,043,000	5,005,000	8,233,000
22,000	3,263,000	5,380,000	8,869,000
23,000	3,488,000	5,763,000	9,523,000
24,000	3,718,000	6,156,000	10,190,000
25,000	3,953,000	6,559,000	10,880,000

A more accurate way of stating the formula for the hysteresis loss is

$$W = \eta f V B_{max.}^{n} \times 10^{-7},$$

where n is a number which varies slightly over different parts of the curve of W and B and also varies for different kinds of iron and steel. For example Ewing and Klaassen found experimentally for a particular specimen of iron that n was nearly 1·55 from B

equal to 1000 to B equal to 2000 and that it was only 1·475 from B equal to 2000 to B equal to 8000. For values of B between 8000 and 14000 they found that for this specimen n was 1·7.

Tables of the 1·5th, 1·55th and 1·6th powers of the numbers representing the induction densities usually wanted in practice are given above.

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CHAPTER II.

Alternating current in an inductive circuit. The calculation of inductance. The E.M.F. required to produce a current i in the inner conductor of a concentric main. Flux of force inside a circular ring of rectangular section. Flux inside a ring of circular cross section. The magnetic analogy of Ohm's law. Reluctance. Inductance formulae for anchor rings. Self inductance of a concentric main. The self inductance of two parallel cylindrical wires. The self inductance of a circuit formed by three equal parallel cylinders, the axes of which lie along the edges of an equilateral prism. Triple concentric main. Minimum self inductance. The repulsive force between two parallel wires carrying equal currents in opposite directions. References.

THE rotation of the armature of an alternator causes a rapid periodic change in the flux of force through its coils and hence the potential difference between the collector rings is a periodic function of the time. We may therefore express the difference of potential mathematically by the expression $f(t)$ where t is the time in seconds, and if T be the time required for the P.D. to go through all its values, *i.e.* if it is the period, then

$$f(t) = f(t+T) = f(t+2T) = \dots .$$

Also, since the north and south poles of the field magnets in actual machines are always made alike, it follows that the wave for the first half period is the same as for the second half with the sign changed, so that

$$f(t) = -f\left(t + \frac{T}{2}\right).$$

By Fourier's formula $f(t)$ may be expressed in this case by the series

$$f(t) = A_1 \sin\left(\frac{2\pi}{T}\, t + \alpha_1\right) + A_3 \sin\left(3\,\frac{2\pi}{T}\, t + \alpha_3\right) + \dots,$$

where A_1, A_3 ... etc. are the amplitudes of the first, third, ... harmonics which can be determined by the usual formulae when $f(t)$ is known. It is customary to express $\frac{2\pi}{T}$ by ω, and we see that ω is the angular velocity of a line rotating $\frac{1}{T}$ times a second. If f be the frequency, *i.e.* the number of times per second that the potential difference goes through all its values, then

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

Now suppose that an alternating P.D., of which the instantaneous value is e, is applied to a circuit of resistance R and self inductance L. Then if i be the instantaneous value of the current in the circuit $-L\frac{di}{dt}$ will be the E.M.F. due to induction, and we have by Ohm's law,

$$i = \frac{e - L\frac{di}{dt}}{R},$$

$$\therefore L\frac{di}{dt} + Ri = e,$$

$$\therefore (LD + R)\, i = A_1 \sin(\omega t + \alpha_1) + A_3 \sin(3\omega t + \alpha_3) + \ldots,$$

$$\therefore i = \frac{A_1}{R + LD}\sin(\omega t + \alpha_1) + \frac{A_3}{R + LD}\sin(3\omega t + \alpha_3) + \ldots\ldots (a).$$

Now $\frac{A_{2n+1}}{R + LD}\sin\{(2n+1)\,\omega t + \alpha_{2n+1}\}$

$$= \frac{A_{2n+1}(R - LD)}{R^2 - L^2D^2}\sin\{(2n+1)\,\omega t + \alpha_{2n+1}\}.$$

Also

$$D \sin\{(2n+1)\,\omega t + \alpha_{2n+1}\} = (2n+1)\,\omega \cos\{(2n+1)\,\omega t + \alpha_{2n+1}\},$$

$$\therefore D^2 \sin\{(2n+1)\,\omega t + \alpha_{2n+1}\} = -(2n+1)^2\omega^2 \sin\{(2n+1)\,\omega t + \alpha_{2n+1}\}.$$

Therefore, using symbolic methods, we can replace

$$D^2 \text{ by } -(2n+1)^2\omega^2.$$

Substituting and simplifying we find that

$$\frac{A_{2n+1}}{R+LD}\sin\{(2n+1)\,\omega t+\alpha_{2n+1}\}$$
$$=\frac{A_{2n+1}\sin\{(2n+1)\,\omega t+\alpha_{2n+1}-\beta_{2n+1}\}}{\sqrt{R^2+(2n+1)^2L^2\omega^2}},$$

where $$\tan\beta_{2n+1}=\frac{(2n+1)\,L\omega}{R}.$$

Also, since the solution of the equation

$$(LD+R)\,i=0$$

is $$i=B\epsilon^{-\frac{R}{L}t},$$

where B is a constant, we see that the complete solution of the equation (a) above is

$$i=\Sigma\frac{A_{2n+1}\sin\{(2n+1)\,\omega t+\alpha_{2n+1}-\beta_{2n+1}\}}{\sqrt{R^2+(2n+1)^2L^2\omega^2}}+B\epsilon^{-\frac{R}{L}t}.$$

In this formula B is a constant which depends on the initial conditions, and ϵ is 2·718 ... the base of Neperian logarithms. Now as $\frac{R}{L}$ is in most cases very great, we see that in these cases the current after a fraction of a second assumes a steady periodic value which is given by

$$i=\frac{A_1\sin(\omega t+\alpha_1-\beta_1)}{\sqrt{R^2+L^2\omega^2}}+\frac{A_3\sin(3\omega t+\alpha_3-\beta_3)}{\sqrt{R^2+3^2L^2\omega^2}}+\ldots.$$

Suppose that the applied P.D. is a simple harmonic function $E\sin\omega t$, and that the switch is closed when t is equal to t_1. Then since the circuit can not receive finite energy in an infinitely short time, we see that the initial value of i must be zero, and the general solution becomes in this case

$$i=\frac{E}{\sqrt{R^2+L^2\omega^2}}\left\{\sin(\omega t-\alpha)-\sin(\omega t_1-\alpha)\,\epsilon^{-\frac{R}{L}(t-t_1)}\right\}\ \ldots\ldots(b),$$

where $$\tan\alpha=\frac{\omega L}{R}.$$

When t is very large the initial disturbance caused by closing the switch will be negligible as the damping factor $\epsilon^{-\frac{R}{L}(t-t_1)}$ will be practically zero, and hence we can write

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \alpha).$$

We see, in this case, that whenever t equals $\frac{\alpha}{\omega} + nT$, then i is zero and

$$\frac{di}{dt} = \frac{E\omega}{\sqrt{R^2 + L^2\omega^2}} = \frac{E \sin\alpha}{L}.$$

Now at the moment of closing the switch

$$E \sin \omega t_1 = Ri + L\frac{di}{dt},$$

therefore
$$\frac{di}{dt} = \frac{E \sin \omega t_1}{L},$$

since i is zero at this instant. Hence, comparing the initial value of $\frac{di}{dt}$ with its value for $t = \frac{\alpha}{\omega} + nT$ after the current has become purely alternating, we see that if $\omega t_1 = \alpha$, *i.e.* if $t_1 = \frac{\alpha}{2\pi}T$, then there will be no initial disturbance, and the current will at once become purely alternating. This can also be seen from equation (b) above, the coefficient of the exponential term vanishing when t_1 is $\frac{\alpha}{\omega}$. It follows from this equation that the initial disturbance is a maximum when

$$t_1 = \frac{\alpha}{\omega} + \frac{T}{4} = \frac{\alpha}{2\pi}T + \frac{T}{4}.$$

It is easy to see also that the maximum value of the current after switching on can never be as great as

$$\frac{2E}{\sqrt{R^2 + L^2\omega^2}}.$$

If t_1 lies in value between $\frac{\alpha}{2\pi}T$ and $\frac{\alpha}{2\pi}T + \frac{T}{2}$ then the maximum values of i on the positive side are smaller initially than

$$\frac{E}{\sqrt{R^2 + L^2\omega^2}},$$

but the maximum values of i on the negative side are greater than

$$\frac{E}{\sqrt{R^2 + L^2\omega^2}}.$$

The number of coulombs that circulate during the first interval T after closing the circuit is

$$\int_{t_1}^{t_1+T} i\,dt = -\frac{L}{R}\frac{E\sin(\omega t_1 - \alpha)}{\sqrt{R^2 + L^2\omega^2}}\left\{1 - \epsilon^{-\frac{R}{L}T}\right\}.$$

Similarly for the second period

$$\int_{t_1+T}^{t_1+2T} i\,dt = -\frac{L}{R}\frac{E\sin(\omega t_1 - \alpha)}{\sqrt{R^2 + L^2\omega^2}}\left\{\epsilon^{-\frac{R}{L}T} - \epsilon^{-\frac{2R}{L}T}\right\}.$$

Hence we see that this number rapidly gets smaller and smaller and the current soon becomes purely alternating.

When the coil has an iron core, the back E.M.F. is no longer proportional to $\frac{di}{dt}$, and the equation to determine the initial disturbance is much more complicated. It is easy to see that the remanence of the iron in the core initially is a principal factor in determining the magnitude of the disturbance on closing the switch. In practice this factor is generally unknown. What happens on closing the switch can be determined experimentally by getting a record of the current with an oscillograph. It is found that, when we have iron in the core, the initial fluctuations of the current are sometimes very large.

The calculation of inductance.

The solution given above for the growth of current in an alternating current circuit shows that in making calculations a knowledge of the inductance of a circuit is quite as important as a knowledge of its resistance. The inductance, as a rule, is not an easy thing to measure and formulae for it can only be given in some very simple cases. We will investigate some of these formulae in this chapter. These formulae are arrived at on the assumptions that the current density is constant across the section of the wires, or that we can consider that it is all on the circumference, or that the wires are of negligible cross section. With high frequencies the first assumption is not admissible, as there is a tendency for the current density to be greater near the circumference of a solid wire than near its axis.

To see what causes the current density to be variable consider the case of a solid cylindrical wire. We may suppose that it is built up of a system of concentric cylindrical shells, and that the current density is constant in any particular shell. When a current starts in an inner cylinder, it produces no inductive effects inside it, as the magnetic force is always zero there (Chapter I.). It produces however inductive effects outside it in exactly the same way as if all the current were concentrated along the axis. The power for these induced currents must be taken from the inducing circuit, and hence the electromagnetic inertia of currents flowing along the cylindrical shells near the axis must be greater than that of cylindrical currents near the circumference.

The E.M.F. required to produce a current i in the inner conductor of a concentric main.

By solving the general equations of the E.M.F. in an electromagnetic field, Clerk Maxwell shows that if e be the P.D. between the ends of the inner conductor, then

$$e = lri + l\left(A + \frac{1}{2}\right)\frac{di}{dt} - \frac{1}{12}\frac{l}{r}\frac{d^2i}{dt^2} + \frac{1}{48}\frac{l}{r^2}\frac{d^3i}{dt^3} - \frac{1}{180}\frac{l}{r^3}\frac{d^4i}{dt^4} + \dots,$$

where r is, in absolute units, the resistance of unit length of the inner conductor, the length of which is l, and A is a constant depending on the return conductor. When the current follows the harmonic law,

$$\frac{d^2i}{dt^2} = -\omega^2 i \text{ and } \frac{d^3i}{dt^3} = -\omega^2\frac{di}{dt},$$

where $\omega = 2\pi f$ and f is the frequency of the alternating current. In this case the above equation becomes

$$e = l\left(r + \frac{1}{12}\frac{\omega^2}{r} - \frac{1}{180}\frac{\omega^4}{r^3} + \dots\right)i$$
$$+ l\left(A + \frac{1}{2} - \frac{1}{48}\frac{\omega^2}{r^2} + \frac{13}{8640}\frac{\omega^4}{r^4} - \dots\right)\frac{di}{dt}.$$

Hence if R_1 and L_1 be the effective resistance and inductance of a length l centimetres of the inner conductor, then

$$\frac{R_1}{l} = r + \frac{1}{12}\frac{\omega^2}{r} - \frac{1}{180}\frac{\omega^4}{r^3} + \dots,$$

$$\frac{L_1}{l} = A + \frac{1}{2} - \frac{1}{48}\frac{\omega^2}{r^2} + \frac{13}{8640}\frac{\omega^4}{r^4} - \dots.$$

In these formulae R_1 and L_1 are in absolute units. To reduce them to ohms and henrys respectively, we have to multiply by 10^{-9}. If R be the resistance of the whole length l of the conductor in ohms, then

$$r = R\frac{10^9}{l}.$$

Hence, if R_1 and L_1 be expressed in ohms and henrys, the formulae can be written as follows:

$$R_1 = R\left\{1 + \frac{1}{12}\left(\frac{l\omega}{R10^9}\right)^2 - \frac{1}{180}\left(\frac{l\omega}{R10^9}\right)^4 + \dots\right\},$$

$$L_1 = l\left\{A + \frac{1}{2} - \frac{1}{48}\left(\frac{l\omega}{R10^9}\right)^2 + \frac{13}{8640}\left(\frac{l\omega}{R10^9}\right)^4 - \dots\right\}10^{-9}.$$

For example, suppose that the conductor was a solid copper rod half an inch in diameter and 1000 feet long; then $l = 30480$ cms. and $R = 0{\cdot}04$ ohm. Suppose also that the frequency was 50, then $\omega = 314{\cdot}2$. Hence

$$\left(\frac{l\omega}{R10^9}\right)^2 = 0{\cdot}0573 \quad \text{and} \quad \left(\frac{l\omega}{R10^9}\right)^4 = 0{\cdot}0033.$$

Therefore $\qquad R_1 = 0{\cdot}04\{1 + 0{\cdot}0048 - 0{\cdot}0000\}$

$$= 0{\cdot}04019 \text{ ohm}.$$

The formulae for R_1 and L_1 show that the resistance and inductance of the cylindrical conductor depend on the frequency of the alternating current flowing in it. This is due to the smaller inductance of the paths for the current near the circumference of the cylinder and the consequent irregular distribution of the current. At very high frequencies the current may be practically confined to a shallow layer near the circumference of the conductor. This phenomenon is generally referred to as the 'skin' effect. It is to be noted that the formulae given above are arrived at on the assumption that the conductor is straight, that the current, and therefore also the applied P.D., follows the harmonic law, and that the lines of force round the conductor are circles.

In the case of the outer conductor, the formulae for the effective resistance and inductance are more complicated than for the inner conductor. With high frequencies the current is nearly confined to the inner surface of the conductor. If we suppose the frequency to be infinite then the current would be confined to the outer surface of the inner conductor and the inner surface of the outer conductor so that no magnetic force would be produced in the interior of the conductors.

If a constant P.D. were suddenly applied to the conductors at one end of a concentric main, the conductors being short circuited at the far end, then the current would attain its steady value first on the outer layer of the inner conductor and on the inner layer of the outer conductor. It might hence be pictured as soaking inwards and outwards respectively. Similarly, when the circuit is broken, the currents in the outer layer of the inner conductor and in the inner layer of the outer conductor will die away the most rapidly.

Heaviside has solved the problem of the propagation of electromagnetic waves along concentric cylinders. He has shown that the energy apparently reaches the inner wire from the outside medium and so the electric current is set up first in the outer layer of the wire and takes time to penetrate into the middle. When an alternating potential difference is applied, the currents in the elementary concentric tubes, of which we may suppose the cylinders to be built up, differ not only in amplitude but also in phase, so that at a certain depth the current may be flowing in an opposite direction to that which it has at the surface. If the conductivity of the conductors were infinite, there could be no current set up in the interior of the wires, and there would be no dissipation of energy. This shows that magnetic induction could not penetrate into an ideally perfect conductor. Somewhat analogous problems will be discussed later in the chapter on eddy currents.

Flux of force inside a circular ring of rectangular section.

Let R be the mean radius of a circular ring of rectangular section, a its radial depth, b its breadth, and N the total number of turns of wire wrapped round it in such a way that we may consider the turns to be in planes passing through the axis of the ring. We will also suppose that everything is symmetrical, so that the whole flux is inside the ring. Now all parts of the core of the ring at the same distance x from its axis are subjected to the same magnetising force. Suppose that the core is of non-magnetic material, then the element of flux $d\phi$ contained in the rectangle $N'PNP'$ (Fig. 12) of breadth dx is $4\pi mbdx$ where m is the polar

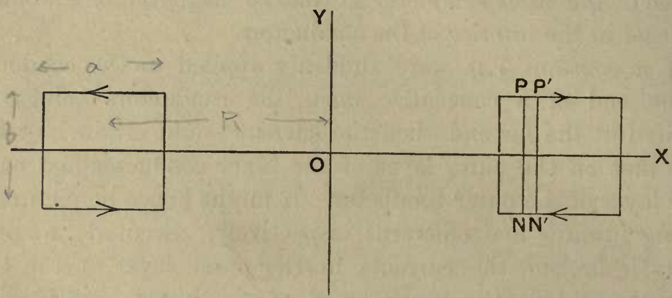

Fig. 12. Flux inside a ring of rectangular cross section.

strength per unit area of the face of the equivalent magnetic shell in this place replacing one turn. Now $i = ml$, where l is the thickness of the shell. Also $lN = 2\pi x$ and hence $m = \frac{N}{2\pi x} i$.

Therefore $$d\phi = 4\pi \cdot \frac{N}{2\pi x} \cdot ibdx$$

$$= 2Ni\frac{b}{x}\,dx.$$

Thus $$\phi = 2Nbi \int_{R-\frac{a}{2}}^{R+\frac{a}{2}} \frac{dx}{x}$$

$$= 2Nbi \log_e \frac{R + \frac{a}{2}}{R - \frac{a}{2}}.$$

If B_m denote the mean flux density over the cross section of this ring, then

$$\begin{aligned} B_m &= \frac{\phi}{ab} \\ &= \frac{2Ni}{a} \log_\epsilon \frac{1 + \frac{a}{2R}}{1 - \frac{a}{2R}} \\ &= \frac{4Ni}{a} \left\{ \frac{a}{2R} + \frac{1}{3} \left(\frac{a}{2R} \right)^3 + \dots \right\} \\ &= \frac{4\pi Ni}{2\pi R} \left\{ 1 + \frac{a^2}{12R^2} + \frac{a^4}{80R^4} + \dots \right\} \\ &= 4\pi ni \left\{ 1 + \frac{a^2}{12R^2} + \frac{a^4}{80R^4} + \dots \right\}, \end{aligned}$$

where n is the number of turns of wire per centimetre length of the mean circumference. If R were infinite in comparison with a, B_m would equal $4\pi ni$, and since, in this case, B_m is independent of a, it is constant whatever the cross section may be and is equal to B. Since in air B equals H, we see that the magnetising force in an infinite solenoid equals $4\pi ni$. If i is in amperes

$$H = \frac{4\pi ni}{10}.$$

When the core of the ring is magnetic, since

$$H = \frac{4\pi Ni}{2\pi x},$$

we see that the magnetising force varies at different points of the cross section and therefore, from the properties of iron, it follows that μ also varies. If however x only vary very little, that is, if $\frac{a}{R}$ be small, then the ratio of B_m to $\frac{4\pi Ni}{l}$, where l is the length of the mean circumference of the ring, gives us an approximate value of μ corresponding to the value of H at the centre of the cross section of the ring.

Owing to the importance of this problem we will work out the case when the section of the ring is circular.

Suppose that the radius of the cross section is a circle of radius a (Fig. 13) then, making the same assumptions as before,

Flux inside a ring of circular cross section.

$$d\phi = 4\pi m \,.\, 2PN \,.\, dx$$

$$= 4Ni \frac{\sqrt{a^2 - (R-x)^2}}{x} dx.$$

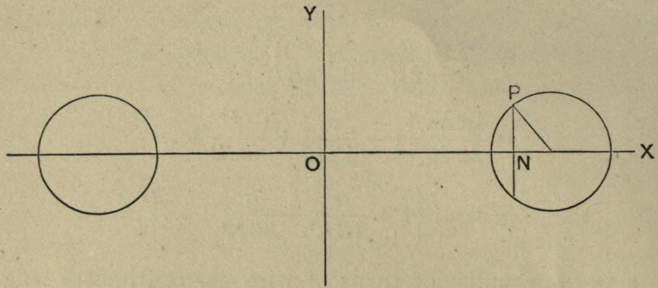

Fig. 13. Flux inside a ring of circular cross section.

Therefore $$\phi = 4Ni \int_{R-a}^{R+a} \frac{\sqrt{a^2 - (R-x)^2}}{x} dx.$$

Writing $x = R - a \sin\theta$, we get

$$\phi = \frac{4Nia^2}{R} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta \left\{1 - \frac{a}{R}\sin\theta\right\}^{-1} d\theta$$

$$= \frac{4\pi Nia^2}{R} \left\{\frac{1}{2} + \frac{1}{2\,.\,4}\frac{a^2}{R^2} + \frac{1\,.\,3}{2\,.\,4\,.\,6}\frac{a^4}{R^4} + \dots\right\}$$

$$= 4\pi Ni \{R - \sqrt{R^2 - a^2}\}.$$

$$\therefore H_m = \frac{4Ni}{a^2} \{R - \sqrt{R^2 - a^2}\}$$

$$= \frac{4\pi Ni}{2\pi R} \left\{1 + \frac{1}{4}\frac{a^2}{R^2} + \frac{1}{8}\frac{a^4}{R^4} + \dots\right\}.$$

In this case also, if $\frac{a}{R}$ be small and i is in amperes, we can assume in practical work that

$$H_m = \frac{4\pi Ni}{10l}.$$

The magnetic analogy of Ohm's law. Reluctance.

If Φ be the flux of induction embraced by an infinitely long solenoid and if n be the number of turns on a length l of it, then, by what we have shown,

$$\Phi = \frac{4\pi n i S}{10 l},$$

where S is the cross section of the solenoid and i denotes the current in amperes. If the core of the solenoid be magnetic,

$$\Phi = \mu \frac{4\pi n i S}{10 l}$$

$$= \frac{\frac{4\pi}{10} n i}{\frac{l}{\mu S}}.$$

We may regard the length l of the solenoid as a tube of induction enclosing a flux Φ which is produced by the magnetising force $\frac{4\pi}{10} ni$. The amount of the flux produced depends on the value of $\frac{l}{\mu S}$, which is called the reluctance of the length l of the solenoid. We see that the reluctance $\mathcal{R}$ of a portion of a tube of induction varies as the length of the portion, and varies inversely as the area of the cross section and as the permeability. Hence we get the magnetic analogy to Ohm's law

$$\text{Flux} = \frac{\text{Magnetomotive Force}}{\text{Reluctance}}.$$

The magnetomotive force $\frac{4\pi}{10} ni$ is the difference of magnetic potential between the two ends of the tube, and it is important to notice that it is independent of the shape of its section. It merely depends on the number of turns and the amperes. Hence electricians often talk about ampere turns per centimetre instead of magnetomotive force.

Since $$H = \frac{4\pi}{10 l} ni$$

we see that one ampere turn per centimetre is 1·257 gausses, and that one gauss is nearly 0·8 ampere turn per centimetre.

Inductance formulae for anchor rings.

Let us imagine an anchor ring wound uniformly with two windings of N_1 and N_2 turns respectively. Let its mean radius be R centimetres and the radius of its cross section a centimetres. Then if we can take a as the radius of the inner coil, since all the flux generated by unit current in the inner coil passes through the outer coil,

$$M = 4\pi N_1 N_2 \{R - \sqrt{R^2 - a^2}\}.$$

When the section is rectangular, of breadth b and radial depth a, we have

$$M = 2bN_1N_2 \log_\epsilon \frac{R + \frac{a}{2}}{R - \frac{a}{2}}.$$

In the above formulae M is in centimetres; to obtain the value in henrys we multiply by 10^{-9}.

To get the self inductance of the rings when both are wound with N turns of wire, put $N_1 = N_2 = N$ in the above formulae. For a ring of circular cross section this gives

$$L = 4\pi N^2 \{R - \sqrt{R^2 - a^2}\}.$$

And for a ring of rectangular cross section

$$L = 2bN^2 \log_\epsilon \frac{R + \frac{a}{2}}{R - \frac{a}{2}}.$$

The most important practical cases, however, in which formulae are required for self inductance, are for concentric cylinders and for cylindrical wires parallel to one another. We will therefore give complete proofs of the formulae for these cases. The method adopted is to calculate the self energy of the circuit when a current I is flowing in it and then equate this expression to $\frac{1}{2}LI^2$.

Self inductance of a concentric main.

A concentric main consists of two hollow copper cylinders, one inside and coaxial with the other, and insulated from it. The outer copper conductor is generally protected by a lead sheath which is insulated from it. The alternating current flows in one cylinder and comes back by the other, and *vice versâ*. Hence in practice the cross sectional areas of the two copper cylinders are made equal to one another as each has to carry the same current.

We will use Kelvin's formula $\Sigma \frac{H^2}{8\pi} dv$ to calculate the self energy of the main. Suppose that the current I flows along the inner copper cylinder and returns by the concentric outer cylinder. Now the magnetic force produced by a cylindrical current sheet at a point outside is the same as if the current were concentrated at its axis, and hence, since the currents in the inner and outer cylinders are equal and flowing in opposite directions, there will be no magnetic force outside the main. Again by page 33 the outside cylindrical current sheet produces no magnetic force inside, and hence, if H be the magnetic force between the two conductors at a distance x from the axis, then

$$H = \frac{2I}{x}.$$

Suppose that the outer and inner radii of the outer and inner cylinders are b_2, b_1, a_2 and a_1 respectively, and that the current density is uniform over their cross sections. Then if H_2 and H_1 are the magnetic forces at points in the copper of the outer and inner conductors, then

$$H_2 = \frac{2I}{x} - \frac{2I}{x}\frac{\pi(x^2 - b_1^2)}{\pi(b_2^2 - b_1^2)} = \frac{2I}{b_2^2 - b_1^2}\left(\frac{b_2^2}{x} - x\right),$$

$$H_1 = \frac{2I}{x}\frac{\pi(x^2 - a_1^2)}{\pi(a_2^2 - a_1^2)} = \frac{2I}{a_2^2 - a_1^2}\left(x - \frac{a_1^2}{x}\right).$$

Hence if L be the self inductance of a length l of the concentric cable,

$$\frac{L}{l} = \frac{2}{I^2}\int_{b_1}^{b_2} \frac{H_2^2}{8\pi} \cdot 2\pi x dx + \frac{2}{I^2}\int_{a_2}^{b_1} \frac{H^2}{8\pi} \cdot 2\pi x dx + \frac{2}{I^2}\int_{a_1}^{a_2} \frac{H_1^2}{8\pi} \cdot 2\pi x dx.$$

Therefore

$$\frac{L}{l} = 2 \log_e \frac{b_1}{a_2} + \frac{2a_1^4}{(a_2^2 - a_1^2)^2} \log_e \frac{a_2}{a_1} + \frac{1}{2}\frac{a_2^2 - 3a_1^2}{a_2^2 - a_1^2}$$
$$+ \frac{2b_2^4}{(b_2^2 - b_1^2)^2} \log_e \frac{b_2}{b_1} - \frac{1}{2}\frac{3b_2^2 - b_1^2}{b_2^2 - b_1^2}.$$

Since in practice the cross sectional areas of the cylinders are equal, we have

$$\pi (b_2^2 - b_1^2) = \pi (a_2^2 - a_1^2).$$

Therefore

$$\frac{L}{l} = 2 \log \frac{b_1}{a_2} + \frac{2}{(a_2^2 - a_1^2)^2} \left\{ a_1^4 \log \frac{a_2}{a_1} + b_2^4 \log \frac{b_2}{b_1} - \frac{1}{2} (a_1^2 + b_2^2)(a_2^2 - a_1^2) \right\}.$$

If the inner cylinder were solid, a_1 would be zero and b_2^2 would equal $b_1^2 + a_2^2$. Using these values in the formula and noting that $a_1^4 \log a_1$ is zero when a_1 is zero we get

$$\frac{L}{l} = 2 \log \frac{b_1}{a_2} + \left(1 + \frac{b_1^2}{a_2^2}\right)^2 \log \left(1 + \frac{a_2^2}{b_1^2}\right) - \frac{b_1^2}{a_2^2} - 1$$

$$= 2 \log \frac{b_1}{a_2} + \frac{1}{2} + \frac{1}{3} \frac{a_2^2}{b_1^2} - \frac{1}{12} \frac{a_2^4}{b_1^4} + \frac{1}{30} \frac{a_2^6}{b_1^6} - \ldots.$$

We see that the least possible value of L is when b_1 equals a_2; in this case

$$L = (4 \log_\epsilon 2 - 2)\, l = 0{\cdot}7726\, l.$$

With very high frequencies the current in the inner conductor would be concentrated along its outer circumference, and the current in the outer conductor would be concentrated along its inner circumference. The magnetic force is thus confined to the space between the two cylinders, and hence

$$L = \frac{2l}{I^2} \int_{a_2}^{b_1} \frac{H^2}{8\pi}\, 2\pi x\, dx$$

$$= 2l \log \frac{b_1}{a_2},$$

where a_2 is the outer radius of the inner cylinder and b_1 is the inner radius of the outer cylinder.

If L be in henrys and l in miles, the general formula is

$$\frac{L}{l} = 0{\cdot}000741 \log_{10} \frac{b_1}{a_2} + 0{\cdot}000161 \left(\frac{1}{2} + \frac{1}{3} \frac{a_2^2}{b_1^2} - \frac{1}{12} \frac{a_2^4}{b_1^4} + \ldots\right) \alpha,$$

where α is a quantity that depends on the frequency. For low frequencies α may be taken equal to unity and for very high frequencies it is zero.

For example, suppose that b_1 and a_2 are 0·406 and 0·192 inch respectively, then the inductance in henrys per mile is given by

$$L = 0{\cdot}000241 + 0{\cdot}000161\,(0{\cdot}5 + 0{\cdot}075 - 0{\cdot}004 + \ldots)\,\alpha$$

$$= 0{\cdot}000241 + 0{\cdot}000092\alpha.$$

The self inductance of two parallel cylindrical wires.

Maxwell calculates the self inductance of two parallel wires by finding the value of $\Sigma \frac{H^2}{8\pi} dv$ and equating it to $\frac{1}{2}LI^2$. It is easier however in this case to calculate the self energy of the circuit from the formula $\frac{1}{2}\Sigma\phi i$ (page 25) where i is the number of tubes of current embraced by and producing the flux ϕ. As the problem is one of considerable importance in electrical engineering it will be useful to give a direct proof of the equivalence of the two formulae for the self energy of a circuit.

Suppose that the circuit can be divided up into n parts and that we can calculate $\frac{1}{2}\Sigma\phi i$ for each of those parts separately. Let $H_1, H_2, \ldots, H_n$ be the forces due to the n portions of the circuit, and let H be the resultant force. Then H equals the sum of the components of $H_1, H_2, \ldots, H_n$, in the direction of H. Let θ_1 denote the angle between H_1 and H, θ_2 the angle between H_2 and H, etc.; then

$$H = H_1 \cos\theta_1 + H_2 \cos\theta_2 + \ldots,$$

therefore

$$H^2 = H_1 H \cos\theta_1 + H_2 H \cos\theta_2 + \ldots.$$

Now to find the volume integral of $H_1 H \cos\theta_1$; take the tube of force corresponding to H_1 and let dS be a section of any equipotential surface by this tube. Then $H_1 dS$ is constant along the tube and equal to ϕ_1. Also $H\cos\theta_1$ is the tangential force along the axis of the tube, and therefore, if ds be an element of the axis of the tube,

$$\int H \cos\theta_1 ds = 4\pi i_1,$$

where i_1 is the total current embraced by the tube. Therefore the volume integral of $H_1 H \cos\theta_1$ for the single tube is

$$\Sigma\Sigma H_1 H \cos\theta_1 dS ds = \phi_1 \Sigma H \cos\theta_1 ds$$
$$= 4\pi\phi_1 i_1.$$

Summing for all the tubes of force, we find that the complete volume integral of $H_1 H \cos\theta_1$ is $4\pi\Sigma\phi_1 i_1$. Hence finally we see that

$$\Sigma\frac{H^2}{8\pi}dv = \tfrac{1}{2}\Sigma\phi_1 i_1 + \tfrac{1}{2}\Sigma\phi_2 i_2 + \ldots + \tfrac{1}{2}\Sigma\phi_n i_n.$$

The self inductance of a circuit may therefore be calculated by the formula

$$L = \frac{1}{I^2}\Sigma\phi i.$$

Consider the case of two parallel hollow cylindrical wires (Fig. 14) and let a_1, a_2 and b_1, b_2 be the inner and outer radii of the two cylinders respectively. We will consider the value of $\Sigma\phi i$ for the lines of force due to the current in each wire separately and then add the results together.

In the cylinder of which the centre is O and radii a_1, a_2 (Fig. 14) we have first to calculate $\Sigma\phi i$ for the lines of force in the substance of the conductor itself, i being the total current embraced by ϕ. We know that the lines of force due to the current in this cylinder are circles and that

$$\phi = \frac{2I}{a_2^2 - a_1^2}\left(x - \frac{a_1^2}{x}\right)dx$$

for a tube of thickness dx and length unity, just as in the inner conductor of a concentric main. Also inside the cylindrical hollow ϕ is zero. The value of ϕi per unit length in the conductor itself is

$$\frac{2I^2}{(a_2^2 - a_1^2)^2}\int_{a_1}^{a_2}\frac{(x^2 - a_1^2)^2}{x}dx,$$

since

$$i = \frac{x^2 - a_1^2}{a_2^2 - a_1^2}I,$$

the current being supposed to be uniform over the cross section. Hence the value of $\Sigma\phi i$ in the conductor itself is

$$\frac{2a_1^4 I^2}{(a_2^2 - a_1^2)^2}\log\frac{a_2}{a_1} + \frac{I^2}{2}\frac{a_2^2 - 3a_1^2}{a_2^2 - a_1^2} \quad\ldots\ldots\ldots\ldots\ldots(1).$$

If d be the distance between the axes of the cylinders, then the value of $\Sigma\phi i$ from $x = a_2$ up to $x = d - b_2$ is

$$\int_{a_2}^{d-b_2} \frac{2I}{x} . I dx = 2I^2 \log \frac{d - b_2}{a_2} \quad \text{...............(2).}$$

Now consider the circular lines of force of radius x due to the current in the cylinder a, where x lies between $d - b_2$ and $d + b_2$. They will embrace all the current in the first cylinder and part of the current in the second. As the current in the second is flowing in the opposite direction to what it is in the first, the part of $\Sigma\phi i$ due to it will be negative. The part of $\Sigma\phi i$ due to values of x between $d - b_2$ and $d + b_2$ and the current in the first cylinder is

$$\int_{d-b_2}^{d+b_2} \frac{2I}{x} . I dx = 2I^2 \log \frac{d + b_2}{d - b_2} \quad \text{...............(3).}$$

To find the linkages with the tubes of current in the second cylinder, divide its section into a series of concentric rings and consider one of them with radius r and thickness dr (Fig. 14).

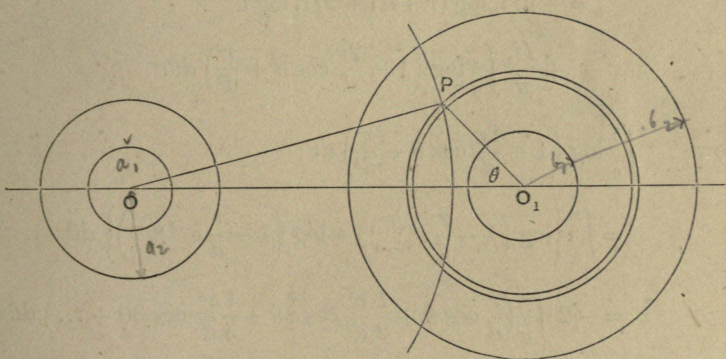

Fig. 14. The self inductance of two parallel hollow cylinders.

Let $OP = x$, $O_1P = r$ and the angle $PO_1O = \theta$. Then

$$x^2 = r^2 + d^2 - 2dr \cos\theta,$$

$$\therefore\ x dx = rd \sin\theta . d\theta$$

when r is kept constant. The current i_1 in the cylinder of which the radius is r and thickness dr is equal to $\frac{2\pi r dr I}{\pi(b_2^2 - b_1^2)}$. Hence the part of $\Sigma\phi i$ due to the current in this cylinder is

$$-\int_{d-r}^{d+r} \frac{\theta}{\pi} \dot{i}_1 \frac{2I}{x} dx,$$

for $\frac{\theta}{\pi} \dot{i}_1$ is the number of tubes of current in this elementary cylinder intercepted by a circle of radius x. Since

$$x dx = rd \sin\theta \,.\, d\theta,$$

this may be written

$$-\frac{I\dot{i}_1}{\pi}\int_0^\pi \frac{2rd\sin\theta\,.\,\theta d\theta}{r^2 + d^2 - 2rd\cos\theta}$$

$$= -\frac{I\dot{i}_1}{\pi}\,\overline{\theta \log(r^2 + d^2 - 2rd\cos\theta)}\Big|_0^\pi$$

$$+ \frac{I\dot{i}_1}{\pi}\int_0^\pi \log(r^2 + d^2 - 2rd\cos\theta)\, d\theta$$

$$= -2I\dot{i}_1 \log(r + d) + 2I\dot{i}_1 \log d$$

$$+ \frac{I\dot{i}_1}{\pi}\int_0^\pi \log\left(1 - \frac{2r}{d}\cos\theta + \frac{r^2}{d^2}\right) d\theta.$$

Now $\int_0^\pi \log\left(1 - \frac{2r}{d}\cos\theta + \frac{r^2}{d^2}\right) d\theta$

$$= \int_0^\pi \left\{\log\left(1 - \frac{r}{d}\epsilon^{\theta\sqrt{-1}}\right) + \log\left(1 - \frac{r}{d}\epsilon^{-\theta\sqrt{-1}}\right)\right\} d\theta$$

$$= -2\int_0^\pi \left(\frac{r}{d}\cos\theta + \frac{1}{2}\frac{r^2}{d^2}\cos 2\theta + \frac{1}{3}\frac{r^3}{d^3}\cos 3\theta + \dots\right) d\theta$$

$$= 0.$$

Therefore up to $d + r$ this elementary cylinder contributes

$$-2I\dot{i}_1 \log(d + r) + 2I\dot{i}_1 \log d \;\dots\dots\dots\dots\dots\dots (\alpha)$$

to $\Sigma\phi i$. It also contributes from $d + r$ up to $d + b_2$

$$-\int_{d+r}^{d+b_2} \dot{i}_1 \frac{2I}{z} dz = -2I\dot{i}_1 \log(d + b_2) + 2I\dot{i}_1 \log(d + r) \dots (\beta).$$

Hence the total contribution by this elementary cylinder is $(\alpha)+(\beta)$, *i.e.*

$$-2Ii_1 \log (d+b_2) + 2Ii_1 \log d.$$

Substituting $\dfrac{2rdrI}{b_2^2-b_1^2}$ for i_1 and integrating from $r=b_1$ to $r=b_2$ we get

$$-2I^2 \log (d+b_2) + 2I^2 \log d \quad\text{..............(4)}$$

as the contribution of the second cylinder.

For values of x greater than $d+b_2$, every tube of force will embrace both the currents I and $-I$ respectively, and hence will contribute nothing to $\Sigma\phi i$.

By adding (1), (2), (3) and (4) together we find that $\Sigma\phi i$ per unit length, where ϕ is due to the current in the first cylinder, is

$$2I^2 \log \frac{d}{a_2} + \frac{2a_1^4 I^2}{(a_2^2-a_1^2)^2} \log \frac{a_2}{a_1} + \frac{I^2}{2}\,\frac{a_2^2-3a_1^2}{a_2^2-a_1^2}.$$

We can write down from symmetry the value of $\Sigma\phi i$ for the second cylinder; and hence adding the two quantities together and dividing by I^2, we find that

$$\frac{L}{l} = 2\log\frac{d^2}{a_2 b_2} + \frac{2a_1^4}{(a_2^2-a_1^2)^2}\log\frac{a_2}{a_1} + \frac{1}{2}\,\frac{a_2^2-3a_1^2}{a_2^2-a_1^2}$$
$$+ \frac{2b_1^4}{(b_2^2-b_1^2)^2}\log\frac{b_2}{b_1} + \frac{1}{2}\,\frac{b_2^2-3b_1^2}{b_2^2-b_1^2}.$$

If we put $\qquad h^2 = a_2^2 - a_1^2$ and $k^2 = b_2^2 - b_1^2$,

this may be written in the form

$$\frac{L}{l} = 2\log_\epsilon \frac{d^2}{a_2 b_2} + \frac{1}{3}\left(\frac{h^2}{a_2^2}+\frac{k^2}{b_2^2}\right) + \frac{1}{12}\left(\frac{h^4}{a_2^4}+\frac{k^4}{b_2^4}\right) + \ldots$$
$$+ \frac{2}{n(n+1)(n+2)}\left(\frac{h^{2n}}{a_2^{2n}}+\frac{k^{2n}}{b_2^{2n}}\right) + \ldots.$$

In calculating this formula we have assumed that the density of the current was constant over the cross sections. With high frequencies the current will be practically confined to a thin layer of metal on the circumference of each of the cylinders, the current flowing in such a way as to produce no magnetic force in the interior of the conductors. The lines of force in this case are the same as the lines of equal potential for two parallel cylinders maintained at potentials $+V$ and $-V$ respectively. Hence as we will see in Chapter V. the formula for the inductance with very high frequencies can be deduced from the corresponding formula for the electrostatic capacity between the two cylinders.

When the currents are uniformly distributed over the cross sections and the cylinders are solid and have equal radii, then noting that $a_1^4 \log a_1$ is zero when a_1 is zero we have

$$L = l\left\{4 \log_\epsilon \frac{d}{a} + 1\right\},$$

where l is the length of either cylinder, a its radius and d is the distance between their axes. If a is small compared to d, then we can write

$$L = l\left\{4 \log_\epsilon \frac{d}{a} + \alpha\right\},$$

where α is a quantity the value of which depends on the frequency. For low frequencies we can take α equal to unity and for very high frequencies α is zero.

In these formulae, if l is in centimetres so also is L. To reduce to henrys we must multiply by 10^{-9}. If l be in statute miles, the following formula will be found useful in practice,

$$L = l\left\{0{\cdot}00148 \log_{10} \frac{d}{a} + 0{\cdot}000161\alpha\right\} \text{ henrys},$$

and for low frequencies we can take α equal to unity. For example if $d = 40$ and $a = 1$ then the self inductance would be 0·00253 henrys per mile at low frequencies.

If the current be uniformly distributed over the cross sections, then

$$L = l\left\{4 \log_\epsilon \frac{d}{a} + 1\right\},$$

and the minimum value of L is obtained when $d = 2a$. In this case

$$\begin{aligned} L &= 2{\cdot}7726\, l + l \\ &= 3{\cdot}7726\, l, \end{aligned}$$

where L and l are in centimetres.

For very high frequencies we shall see in Chapter V. that

$$L = 4l \log_\epsilon \left(\frac{d + \sqrt{d^2 - 4a^2}}{2a}\right).$$

Hence when d is equal to $2a$, L is zero and the currents simply concentrate themselves along the line of contact.

The self inductance of a circuit formed by three equal parallel cylinders the axes of which lie along the edges of an equilateral prism.

We may now consider three mains connected at their far ends, and suppose that a current i_1 flows into No. 1 main, and that currents i_2 and i_3 come back by No. 2 and No. 3 mains respectively, so that

$$i_1 = i_2 + i_3.$$

Let the radius of each cylinder be a, let d be the distance between their axes and suppose that the currents are evenly distributed over their cross sections. Forming $\Sigma\phi i$ as in the last problem, and equating it to $\frac{L}{l} i_1^2$, where l is the length of each cylinder, we get

$$\begin{aligned}\frac{L}{l} i_1^2 = 2i_1^2 \log \frac{d}{a} + \tfrac{1}{2} i_1^2 \\ + 2i_2^2 \log \frac{d}{a} + \tfrac{1}{2} i_2^2 \\ + 2i_3^2 \log \frac{d}{a} + \tfrac{1}{2} i_3^2,\end{aligned}$$

therefore $$L = l\left(4 \log \frac{d}{a} + 1\right)\left(1 - \frac{i_2 i_3}{i_1^2}\right).$$

We see that L has a maximum value $l\left(4 \log \frac{d}{a} + 1\right)$ when either i_2 or i_3 is zero, and a minimum value $\frac{3}{4} l \left(4 \log \frac{d}{a} + 1\right)$ when

$$i_2 = i_3 = \tfrac{1}{2} i_1.$$

Triple concentric main.

Consider the case of three hollow coaxial cylinders of negligible thickness with radii a, b and c respectively, a being that of the smallest. Now if a current i_1 flow in the inner, and return by the two outer cylinders, the currents in which are i_2 and i_3 respectively, and if $L_{1\cdot 23}$ denote the self inductance of the system in this case, then

$$\tfrac{1}{2} L_{1\cdot 23} i_1^2 = \frac{l}{8\pi}\left[\int_a^b \frac{4i_1^2}{x^2} . 2\pi x dx + \int_b^c \frac{4(i_1 - i_2)^2}{x^2} . 2\pi x dx\right]$$

$$= i_1^2 l \log\frac{b}{a} + l(i_1 - i_2)^2 \log\frac{c}{b}.$$

Also $$i_1 = i_2 + i_3.$$

Therefore $$L_{1\cdot 23} = 2l \log\frac{b}{a} + 2l\left(\frac{i_3}{i_1}\right)^2 \log\frac{c}{b}.$$

Similarly $$L_{2\cdot 31} = 2l\left(\frac{i_1}{i_2}\right)^2 \log\frac{b}{a} + 2l\left(\frac{i_3}{i_2}\right)^2 \log\frac{c}{b},$$

and $$L_{3\cdot 12} = 2l\left(\frac{i_1}{i_3}\right)^2 \log\frac{b}{a} + 2l \log\frac{c}{b}.$$

The above formulae could also be calculated as follows:

$$L_{1\cdot 23} i_1^2 = \Sigma\phi i$$

$$= li_1 \int_a^b \frac{2i_1}{x} dx + l(i_1 - i_2)\int_b^c \frac{2i_1}{x} dx - l(i_1 - i_2)\int_b^c \frac{2i_2}{x} dx.$$

Thus $$L_{1\cdot 23} = 2l \log\frac{b}{a} + 2l\left(\frac{i_3}{i_1}\right)^2 \log\frac{c}{b},$$

which is the same result as before.

Minimum self inductance.

In general testing work it is often important to arrange a circuit so that its self inductance may be as small as possible. The formula $\frac{1}{I^2}\Sigma\phi i$ for the self inductance shows us that we have to make $\Sigma\phi i$ a minimum. If the flux ϕ contained in a tube of force embrace both the going and return current, then ϕi for this tube will be zero. Hence in order to make the self inductance as small as possible, it is necessary that the going and return currents should be very close together. When we consider also that the tubular filaments of current in the conductors themselves add an appreciable amount to the total sum of ϕi, we see that the conductors should be flat strips of metal separated from one another by the thinnest possible insulating material. In this case however the circuit possesses considerable electrostatic capacity.

The repulsive force between two parallel wires carrying equal currents in opposite directions.

From their equivalence to magnetic shells, we see that two wires carrying currents flowing in opposite directions repel one another. Let F be the force of repulsion between portions of length l of two parallel wires carrying currents i and $-i$ respectively at a distance x apart. If the wires are cylindrical, then we have shown in Chapter I. that the force exerted by the tubes of flow in one of them at external points is the same as if the current were concentrated along its axis. Similarly it is easy to show that the resultant force on the other cylinder is the same as if the current in the other cylinder were concentrated along its axis.

Therefore
$$F = \frac{2i}{x} \times il$$
$$= \frac{2li^2}{x}.$$

This formula was given by Ampère.

The same result can be deduced from the formula for the self inductance of two parallel wires forming part of one circuit. For suppose that we keep the current i constant while x becomes $x + dx$, then

the rate of working of the E.M.F. per unit length $= i\dfrac{d}{dt}(Li)$
$$= i^2 \frac{dL}{dt},$$
the rate of working against the external mechanical forces
$$= F\frac{dx}{dt},$$
and the rate of increase of the magnetic energy of the system
$$= \frac{d}{dt}(\tfrac{1}{2}Li^2),$$

therefore
$$i^2\frac{dL}{dt} = F\frac{dx}{dt} + \frac{d}{dt}(\tfrac{1}{2}Li^2),$$
hence
$$F = \tfrac{1}{2}i^2\frac{dL}{dx}$$
$$= \tfrac{1}{2}i^2\frac{d}{dx}\left(4l\log\frac{x}{a} + l\right)$$
$$= \frac{2li^2}{x}.$$

If the two wires form part of a circuit, there will also be a tension along each of them due to the action of the magnetic field on the conductors which must close the circuit. If $\frac{1}{2}T$ denote the tension along each conductor, then, proceeding as above, we get

$$T = \tfrac{1}{2} i^2 \frac{dL}{dl}$$

$$= i^2 \left(2 \log \frac{x}{a} + \tfrac{1}{2}\right),$$

where a is the radius of either wire and x the distance between them.

For example, suppose that i is 1000 amperes or 100 C.G.S. units, and that $\frac{x}{a}$ is 10, then

$$T = 100^2 (2 \log_\epsilon 10 + 0{\cdot}5)$$
$$= 100^2 \times 5{\cdot}105$$
$$= 51{,}050 \text{ dynes}$$
$$= 52 \text{ grams weight nearly.}$$

Hence the pull along each conductor will be only equal to the weight of 26 grams.

If the wires are surrounded by a magnetic medium, the forces will be much greater.

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CHAPTER III.

Effective values of complex currents and pressures. Mean value of an alternating current or pressure. Graphical methods of finding mean square values. Equivolt curves. Choking coil currents. Condenser currents. Effect of altering the resistance of a circuit on the form of the current wave. A simple sine wave of E.M.F. produces the maximum current in an inductive coil and the minimum current in a condenser. Resonance. Resonance of currents. Numerical examples. Method of neutralising the inductive effect of a choking coil. Method of neutralising the capacity effect of a shunted condenser. Comparison of inductances by means of a voltmeter. References.

THE frequency of the alternating currents employed in practice varies between 25 and 100 cycles per second. For lighting purposes it is probable that 50 and for power transmission purposes that 25 will become the standard frequency. Now the instruments used to measure alternating amperes and volts are comparatively speaking sluggish in their action, and so their indications do not give the instantaneous values. Consider for example a Siemens electro-dynamometer. The effect of a current i through the fixed coil acting on the movable coil which is traversed by the same current is to produce a couple of which the instantaneous values are proportional to i^2. The couple will therefore have the same sign in whichever direction the current passes through the dynamometer, and hence a deflection will be produced by an alternating current. If the periodic time of the fluctuations be small compared with the periodic time of the moving coil, then the coil is insensible to the fluctuations of i^2, and the mean torque on the moving coil, when its plane is perpendicular to the plane of the fixed coil, can be measured by the torsion of the spiral spring required to bring the coil back to this position. This torsion is measured by the angle through which a pointer has to be turned. Hence if R be the reading,

$$k^2R = \text{the mean value of } i^2,$$

where k^2 is a constant, determined by experiments with direct current. When i is constant, $k\sqrt{R}$ gives us its value in amperes; when i is variable, $k\sqrt{R}$ gives us the square root of its mean square in the same units. The instrument therefore tells us the square root of the mean square (which is sometimes called the R.M.S.), or the effective value, of the current.

When i varies harmonically with the time, it is easy to find the effective value of the current in terms of its maximum value. For example, suppose that

$$i = I \sin \omega t,$$

where $\omega = 2\pi f$ and f is the frequency, then

$$\begin{aligned} i^2 &= I^2 \sin^2 \omega t \\ &= \frac{I^2}{2} - \frac{I^2}{2} \cos 2\omega t. \end{aligned}$$

Now the mean value of $\cos 2\omega t$ over a whole period is zero, for if we plot out the cosine curve, a glance will show that for every positive ordinate in the first and fourth quarter periods there is an equal negative ordinate in the second and third quarter periods, and hence, if we add them all together and divide by the number of them in order to get their mean value, the result will be zero. Therefore the mean value of i^2 is $\frac{I^2}{2}$. If we call the effective value of the current A, then

$$A = \frac{I}{\sqrt{2}} = 0{\cdot}7071 I.$$

Therefore the effective value in this case is about 71 per cent. of the maximum value.

In like manner we can show that those alternating current voltmeters, of which the readings depend on the expansion of a heated wire, or on the electromagnetic attractions and repulsions of moving coils, or on the electrostatic repulsions and attractions of movable segments, give us the square root of the mean square values of the voltages which we measure with them. If the instantaneous value e of the potential difference between the voltmeter terminals be given by

$$e = E \sin \omega t,$$

then, as before, its effective value V is given by

$$V = \frac{1}{\sqrt{2}} E = 0{\cdot}7071 E.$$

Effective values of complex currents and pressures.

If i do not vary harmonically, then by Fourier's Theorem it may be written as follows:

$$i = I_1 \sin(\omega t - \alpha_1) + I_3 \sin(3\omega t - \alpha_3) + \ldots \quad \ldots (1).$$

Hence squaring and taking mean values, we find that the effective value A of the current is given by

$$A^2 = \tfrac{1}{2} I_1^2 + \tfrac{1}{2} I_3^2 + \tfrac{1}{2} I_5^2 + \ldots.$$

If R be the ohmic resistance of the conductor in which i is flowing, then

$$RA^2 = RA_1^2 + RA_3^2 + RA_5^2 + \ldots,$$

where A_1, A_3, A_5 ... are the effective values of the harmonics of the current. We thus see that each harmonic component produces its own heating effect on the conductor in exactly the same way as if all the other harmonics were absent.

If the alternating current i have a direct current component C, then

$$i = C + I_1 \sin(\omega t - \alpha_1) + I_3 \sin(3\omega t - \alpha_3) + \ldots.$$

$$\therefore \; A^2 = C^2 + A_1^2 + A_3^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Hence, to find the heating effect on a conductor produced by a combined direct and alternating current, we calculate the heating effect produced by each separately and add up the results. For example a complex current consisting of 10 amperes of direct current and 10 effective amperes of alternating current would only heat a conductor as much as a direct or an alternating current of 14·14 amperes, and an alternating current ammeter placed in the circuit would only read 14·14 amperes. We shall see later the importance of this result in the theory of rotary converters, and it is utilised in systems of polycyclic distribution.

If the potential difference applied to the voltmeter terminals be given by

$$e = P + E_1 \sin(\omega t - \alpha_1) + E_3 \sin(3\omega t - \alpha_3) + \ldots,$$

then $$V^2 = P^2 + V_1^2 + V_3^2 + \ldots \qquad \text{(3)},$$

where $$V_1^2 = \tfrac{1}{2}E_1^2, \quad V_3^2 = \tfrac{1}{2}E_3^2 \ldots.$$

Hence if we put 100 volts direct current pressure in series with 100 volts alternating current pressure, the reading on a voltmeter would be $100\sqrt{2}$, *i.e.* 141·4 volts, no matter what was the shape of the wave of alternating pressure.

Mean value of an alternating current or pressure.

Although in a few cases the negative half of a wave of alternating current or pressure is not similar to the positive half, as for example when we have aluminium electrodes or electric arcs between metals in the circuit, yet in general they are exactly alike, so that their mean value over a whole period is zero. In a few cases it is useful to know their mean value over half a period. The mean value of $E \sin \omega t$, for example, over half a period, starting from t equal to zero, is

$$\frac{2}{T}\int_0^{\frac{T}{2}} E \sin \omega t dt = \frac{2}{\pi} E = 0{\cdot}6366E;$$

this is less than $0{\cdot}7071E$, which is the effective value. The root mean square value of a variable quantity is always greater than its mean value. This follows at once from the algebraical theorem that

$$\left\{\frac{y_1^2 + y_2^2 + y_3^2 + \ldots + y_n^2}{n}\right\}^{\frac{1}{2}} \text{ is greater than } \frac{y_1 + y_2 + \ldots + y_n}{n},$$

except when all the n quantities are equal to one another—when the two expressions are equal to each other.

Graphical methods of finding mean square values.

In practice the readings on electrostatic, hot wire and moving coil instruments give us at once the square root of the mean square of the alternating currents and pressures. We can also by oscillographs, ondographs, etc. find the shape of the current and pressure waves. It is important to test whether the root mean square pressure or current got from these waves agrees with the voltmeter or ammeter reading. There are several graphical methods of doing this; the following method is particularly convenient when the curve is drawn on sectional paper.

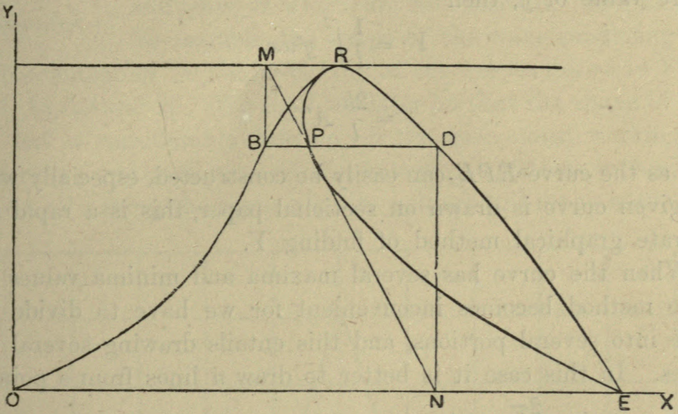

Fig. 15. Graphical construction for finding effective values.

$$Y^2 = \frac{2h}{l}(\text{area } RPED).$$

In Fig. 15 suppose that $OBRE$ is the positive part of any wave curve and that the negative half is exactly similar. Draw a line through R, the highest point of the curve, parallel to OX, and draw any chord BD parallel to it. Draw BM and DN perpendicular to RM and OX respectively and join MN, cutting BD in P. Construct RPE the curve locus of P. Then the volume of the solid generated by the revolution of ORE about OX

$$= \int_0^l \pi y^2 dx.$$

Also $$= \int_0^h 2\pi y \,.\, BD dy,$$

where l is the breadth and h the height of the curve.

$$\therefore \int_0^l y^2 dx = 2\int_0^h y \,.\, BD dy.$$

But $$y\,.\,BD = h\,.\,PD.$$

$$\therefore \int_0^l y^2 dx = 2\int_0^h h\,.\,PD dy$$
$$= 2hA',$$

where A' is the area of the curve $RPED$. If Y^2 be the mean square value of y, then

$$Y^2 = \frac{1}{l}\int_0^l y^2 dx$$
$$= \frac{2h}{l}A'.$$

Now as the curve RPE can easily be constructed, especially when the given curve is drawn on sectional paper, this is a rapid and accurate graphical method of finding Y.

When the curve has several maxima and minima values the above method becomes inconvenient for we have to divide the curve into several portions, and this entails drawing several loci curves. In this case it is better to draw n lines from a point O making angles $\frac{2\pi}{n}$ with one another, and measure along these lines lengths OP_1, OP_2 ... equal in length to the value of n equidistant ordinates of the given curve. Then the area A of the curve drawn through P_1, P_2 ... will be given by

$$A = \tfrac{1}{2}\int_0^{2\pi} r^2 d\theta$$
$$= \pi Y^2.$$

Therefore $$Y = \sqrt{\frac{A}{\pi}}.$$

A useful formula for Y can be found by equating the two well-known expressions for the volume of the solid of revolution formed by rotating the given curve round its time axis,

$$\pi\int_0^l y^2 dx = 2\pi\bar{y}A,$$

and $$Y^2 = 2y_m\bar{y} \quad\text{......................(4).}$$

Y is the R.M.S. value of y, while y_m is its mean value, which equals $\frac{A}{l}$, and $\bar{y}$ is the height of the centre of gravity of the plane figure

ORE above *OX*, which can easily be found by the ordinary methods. When there is only one maximum point the first method of finding y is the most accurate in practice, but formula (4) is a useful one.

Equivolt curves.

It is instructive to draw curves the mean square values of the ordinates of which are all constant. In practice for example, the shape of the wave producing an effective voltage of 50 might be any of the curves drawn in Figs. 16, 17, 18, 19 and 20. We shall see later on that the shape of the wave has a considerable bearing on the economical working of transformers and motors—and the following equations to families of waves all having the same effective voltage will repay study.

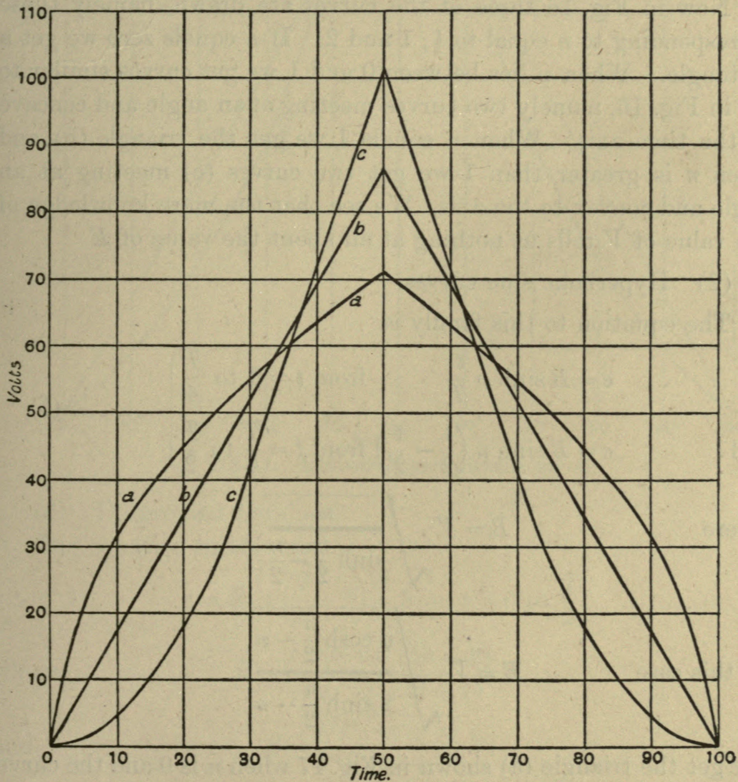

Fig. 16. Equivolt curves. The effective voltage of each curve is 50.

(1) The equations to the curves shown in Fig. 16 are

$$\left.\begin{aligned} e &= E\left(\frac{4}{T}\right)^n t^n \qquad \text{from } t=0 \text{ to } t=\frac{T}{4} \\ e &= E\left(\frac{4}{T}\right)^n \left(\frac{T}{2}-t\right)^n \text{ from } t=\frac{T}{4} \text{ to } t=\frac{T}{2} \end{aligned}\right\} \quad \ldots\ldots(a),$$

and the negative half of the wave is supposed to be similar. In these equations e is the instantaneous value of the E.M.F., t is the time in seconds and T is the period of the alternating current. It follows that

$$E = V\sqrt{2n+1},$$

where E is the maximum value of e and V is its effective value. Hence for a given V the maximum value E increases with n.

Now in Fig. 16 three of the curves are drawn, namely those corresponding to n equal to $\frac{1}{2}$, 1 and 2. If n equals zero we get a rectangle. When n lies between 0 and 1 we get curves similar to (a) in Fig. 16, namely two curves meeting at an angle and concave to the time axis. When n equals 1 we get the triangle (b), and when n is greater than 1 we get two curves (c) meeting at an angle and convex to the axis. We see that the mere knowledge of the value of V tells us nothing at all about the value of E.

(2) Hyperbolic sine curves.

The equation to this family is

$$e = B\sinh n\frac{t}{T} \qquad \text{from } t=0 \text{ to } \frac{T}{4}$$

and

$$e = B\sinh n\left(\frac{1}{2}-\frac{t}{T}\right) \text{ from } t=\frac{T}{4} \text{ to } \frac{T}{2} \quad \ldots\ldots\ldots(b),$$

where

$$B = V\sqrt{\frac{n}{\sinh\frac{n}{2}-\frac{n}{2}}}.$$

In this case

$$E = V\sqrt{\frac{n\cosh\frac{n}{2}-n}{2\sinh\frac{n}{2}-n}}.$$

We get the triangle (a) shown in Fig. 17 when n is 0 and the curve (b) is for n equal to 10.

Tables of sinh θ and cosh θ are given at the end of Chapter XVI.

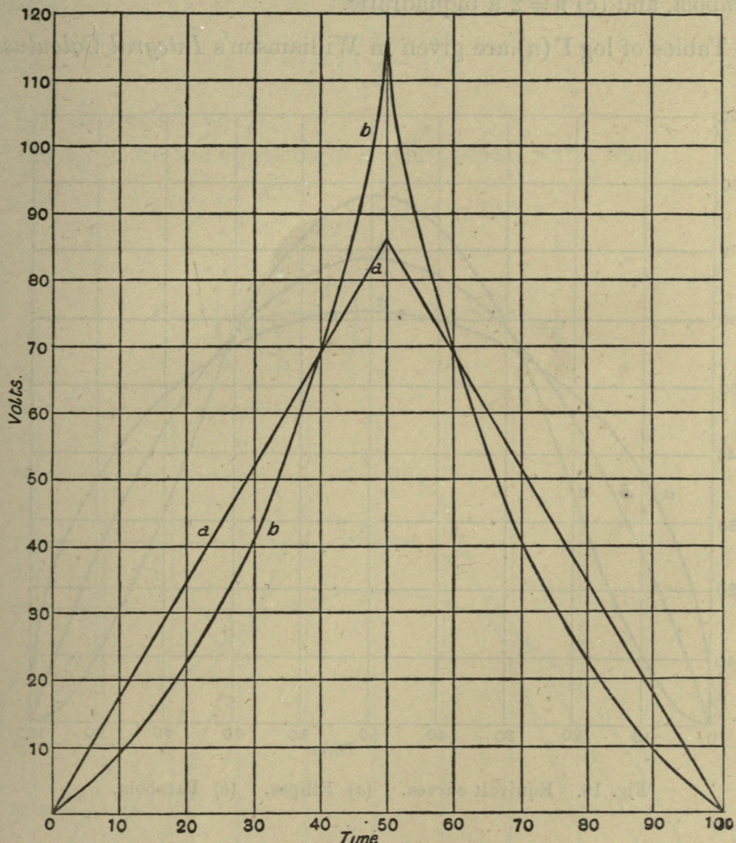

Fig. 17. Equivolt hyperbolic sine curves.

(3) Ellipse and Parabola.

The equation in this case is

$$e = B\left\{t\left(\frac{T}{2} - t\right)\right\}^n \text{ from } t = 0 \text{ to } \frac{T}{2} \quad \ldots\ldots\ldots\ldots(c),$$

where

$$B = \left(\frac{2}{T}\right)^{2n} \frac{\sqrt{\Gamma(4n+2)}}{\Gamma(2n+1)} V,$$

and

$$E = \frac{\sqrt{\Gamma(4n+2)}}{4^n \Gamma(2n+1)} V.$$

Note that $\Gamma(n+1) = n\Gamma(n)$, $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The curves in Fig. 18 are (a) $n=\frac{1}{2}$ an ellipse, (b) $n=1$ a parabola, and (c) $n=2$ a biquadratic.

Tables of $\log \Gamma(n)$ are given in Williamson's *Integral Calculus*.

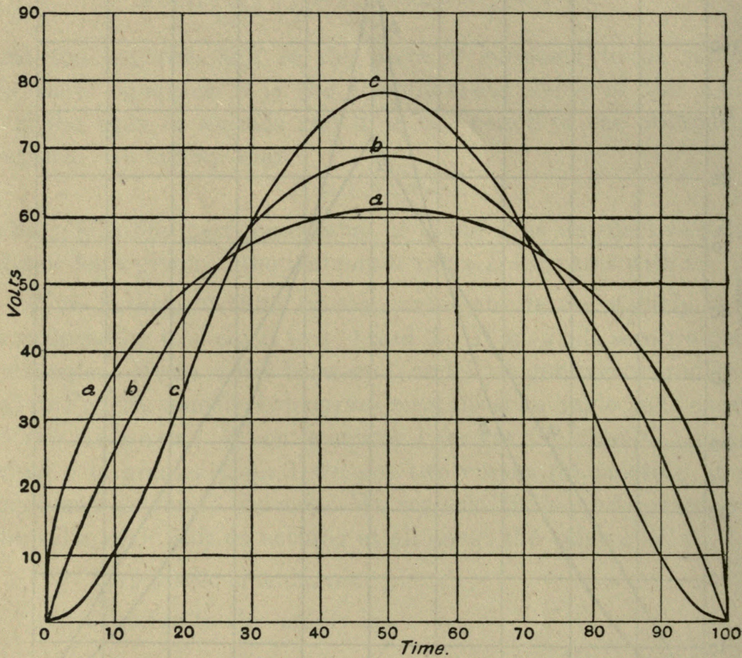

Fig. 18. Equivolt curves. (*a*) Ellipse. (*b*) Parabola.

(4) Sine curves.

The equation to the E.M.F. is

$$e = E \sin^n \frac{2\pi t}{T} \text{ from } t = 0 \text{ to } \frac{T}{2} \ldots\ldots\ldots\ldots\ldots(d).$$

$$\therefore E = \sqrt{\frac{\pi^{\frac{1}{2}}\,\Gamma(n+1)}{\Gamma\left(\frac{2n+1}{2}\right)}}\,V.$$

When $n=0$ we get a rectangle, $n=\frac{1}{2}$ we get the curve (a) in Fig. 19, when $n=1$ the sine curve (b), and $n=2$ the curve (c).

It will be noted that all the waves figured above are symmetrical waves, *i.e.* those in which

$$f(t) = f\left(\frac{T}{2} - t\right).$$

(5) Distorted waves.

If $e = f(t)$ be the equation of a symmetrical wave, then

$$\left.\begin{aligned} e &= f\left(\frac{T}{4}\cdot\frac{t}{\tau}\right) && \text{from } t = 0 \text{ to } \tau \\ e &= f\left(\frac{T}{4}\cdot\frac{\frac{T}{2} - t}{\frac{T}{2} - \tau}\right) && \text{from } t = \tau \text{ to } \frac{T}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(e)$$

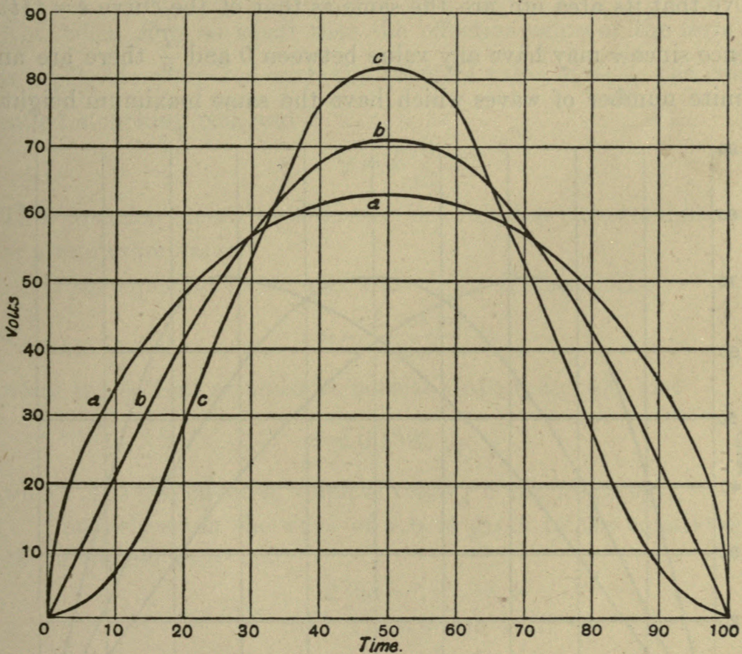

Fig. 19. Equivolt sine curves.

represents a distorted wave of E.M.F. which has the same maximum height, the same R.M.S. height, the same breadth, and the same area as the original wave.

This is easily proved. For example, if V be the R.M.S. of the values of e given by the above equations, then

$$\frac{T}{2}V^2 = \int_0^{\tau}\left\{f\left(\frac{T}{4}\cdot\frac{t}{\tau}\right)\right\}^2 dt + \int_{\tau}^{\frac{T}{2}}\left\{f\left(\frac{T}{4}\cdot\frac{\frac{T}{2}-t}{\frac{T}{2}-\tau}\right)\right\}^2 dt$$

$$= \frac{4\tau}{T}\int_0^{\frac{T}{4}}\{f(t)\}^2\,dt + \frac{4\left(\frac{T}{2}-\tau\right)}{T}\int_0^{\frac{T}{4}}\{f(t)\}^2\,dt$$

$$= \int_0^{\frac{T}{2}}\{f(t)\}^2\,dt.$$

Hence V is independent of the values of τ. Similarly we can prove that its area etc. are the same as that of the curve $e = f(t)$. Hence since τ may have any value between 0 and $\frac{T}{2}$ there are an infinite number of waves which have the same maximum height,

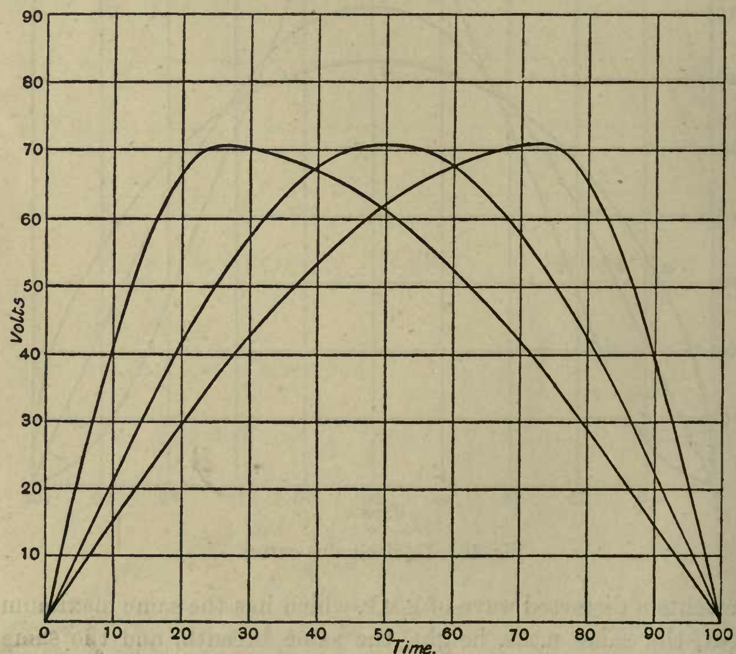

Fig. 20. Equivolt sine curves of equal height.

the same area and the same R.M.S. height as the original symmetrical wave. In Fig. 20 the middle curve shown is the sine curve and the others are distorted members of the same family.

We will refer to the family of waves given by the equations (e) above as a family of waves of equal height. It must be borne in mind, however, that there is an infinite number of families of waves of equal height. For example (a) in Fig. 16 and the sine wave (b) in Fig. 19 are the symmetrical members of two families of waves whose maximum heights are equal.

Choking coil currents.

We have seen (page 41) that the equation connecting the applied P.D. and the current in an inductive coil is

$$e = Ri + L\frac{di}{dt}.$$

Now when R is so small that the effective value of the term Ri is negligible compared with the effective value of $L\frac{di}{dt}$, the coil is called a choking coil, and

$$e = L\frac{di}{dt}.$$

This equation shows that when e is zero, i has either a maximum or a minimum value.

If the equation to the P.D. wave is $e = E \sin \omega t$, then

$$i = -\frac{E \cos \omega t}{L\omega}$$

when the current assumes its normal shape (page 43), and

$$A = 0{\cdot}1591\frac{V}{fL}$$

where A is the effective value of i and f is the frequency.

Similarly when the wave of P.D. is given by the equations (a) above, we can show that

$$A = \left\{\frac{2(2n+1)}{(n+2)(2n+3)}\right\}^{\frac{1}{2}}\frac{V}{4fL}.$$

As n increases from zero, A increases until it attains its maximum value, for

$$n = \frac{\sqrt{6}-1}{2} = 0{\cdot}7247;$$

it then diminishes for greater values of n.

The maximum value of $A = 0{\cdot}1589 \dfrac{V}{fL}$, and E is then $1{\cdot}565V$.

And for a sine curve $A = 0{\cdot}1591 \dfrac{V}{fL}$, and $E = 1{\cdot}414V$.

It will be seen that the sine wave of P.D. produces a larger choking coil current than any of the first family of waves considered.

For the hyperbolic sine curves,

$$A = \frac{V}{fL}\left\{\frac{2n - 6\sinh\frac{n}{2} + n\cosh\frac{n}{2}}{n^2\left(2\sinh\frac{n}{2} - n\right)}\right\}^{\frac{1}{2}}.$$

Condenser currents.

If K be the capacity in farads (see Chapter IV.) of a condenser of which the terminals have a P.D. of v volts, then

$$q = Kv,$$

where q is the number of coulombs in the condenser.

Hence if i be the current flowing into the condenser,

$$i = \frac{dq}{dt}$$

$$= K\frac{dv}{dt}.$$

If A be the effective value of the condenser current when waves of P.D. similar to those illustrated above are applied to the terminals of the condenser, then

	Minimum value.
(a) $A = 4n\sqrt{\dfrac{2n+1}{2n-1}}\,fKV$;	$6{\cdot}661fKV$.
(b) $A = n\left\{\dfrac{\sinh\frac{n}{2} + \frac{n}{2}}{\sinh\frac{n}{2} - \frac{n}{2}}\right\}^{\frac{1}{2}} fKV$;	
(c) $A = 2\sqrt{\dfrac{2n(4n+1)}{2n-1}}\,fKV$;	$6{\cdot}293fKV$.
(d) $A = \dfrac{2\pi n}{\sqrt{2n-1}}\,fKV$;	$6{\cdot}283fKV$ (sine curve).

It will be seen that the sine wave produces the least effective value of the condenser current of any of the waves we have considered. It is also not difficult to show that of all E.M.F. waves of equal height (Fig. 20), applied to choking coils and condensers, the symmetrical wave produces the maximum effective current in the choking coil, and the minimum effective current in the condenser.

Effect of altering the resistance of a circuit on the form of the current wave.

We have seen (page 42) that the solution for the current in an inductive coil is

$$i=\frac{E_1 \sin(\omega t+\alpha_1-\beta_1)}{\sqrt{R^2+L^2\omega^2}}+\frac{E_3 \sin(3\omega t+\alpha_3-\beta_3)}{\sqrt{R^2+L^2(3\omega)^2}}+\dots \qquad \dots\dots(5)$$

$$=\frac{1}{\sqrt{R^2+L^2\omega^2}}\left\{E_1 \sin(\omega t+\alpha_1-\beta_1)\right.$$

$$\left.+\sqrt{\frac{R^2+L^2\omega^2}{R^2+L^2(3\omega)^2}}\,E_3 \sin(3\omega t+\alpha_3-\beta_3)+\dots\right\}.$$

When R is very large β_1, β_3 ... are all very small, and the curve i is practically the same as the curve e on a diminished scale. The more we diminish R, the smaller do the amplitudes of the higher harmonics become as compared with the amplitude of the fundamental harmonic. Hence by diminishing R in an inductive circuit we make i more like a simple sine curve.

If we have resistance in series with a condenser in a circuit, then

$$e=Ri+\frac{\int i\,dt}{K}.$$

And

$$i=\frac{E_1 \sin(\omega t+\alpha_1+\gamma_1)}{\sqrt{R^2+\frac{1}{K^2\omega^2}}}+\frac{E_3 \sin(3\omega t+\alpha_3+\gamma_3)}{\sqrt{R^2+\frac{1}{K^2(3\omega)^2}}}+\dots.$$

Hence, proceeding as before, we deduce the following working rule. Diminishing the resistance in a condenser circuit makes the current wave less like a sine wave.

The equation of the current in an inductive circuit is

A simple sine wave of E.M.F. produces the maximum current in an inductive coil and the minimum current in a condenser.

$$e = Ri + L\frac{di}{dt}.$$

Thus $$V^2 = R^2A^2 + \frac{1}{T}\int_0^T \left(L\frac{di}{dt}\right)^2 dt,$$

since $\int_0^T 2RLi\frac{di}{dt}dt$ vanishes.

Now from page 42,

$$A^2 = \frac{V_1^2}{R^2 + L^2\omega^2} + \frac{V_3^2}{R^2 + L^2(3\omega)^2} + \dots.$$

And

$$\frac{1}{T}\int_0^T \left(L\frac{di}{dt}\right)^2 dt = L^2\omega^2 \left\{\frac{V_1^2}{R^2 + L^2\omega^2} + \frac{3^2V_3^2}{R^2 + L^2(3\omega)^2} + \dots\right\}$$

$$= \alpha^2\omega^2L^2A^2,$$

where $$\alpha^2 = \frac{\dfrac{V_1^2}{R^2 + L^2\omega^2} + \dfrac{3^2V_3^2}{R^2 + L^2(3\omega)^2} + \dots}{\dfrac{V_1^2}{R^2 + L^2\omega^2} + \dfrac{V_3^2}{R^2 + L^2(3\omega)^2} + \dots}.$$

Now the numerator of this fraction is greater than the denominator except when

$$V_3 = V_5 = \dots = 0.$$

Hence α has its minimum value unity when the applied E.M.F. is sine-shaped.

Also $$A^2 = \frac{V^2}{R^2 + \alpha^2L^2\omega^2} \quad\text{.......................(6)},$$

and the denominator has its minimum value, and therefore A has its maximum value, when α is unity.

Therefore the sine wave produces the maximum effective current in an inductive coil.

Similarly for a condenser circuit,

$$A^2 = \frac{V^2}{R^2 + \dfrac{1}{\beta^2K^2\omega^2}} \quad\text{.....................(7)},$$

and β has its minimum value unity when the wave is sine-shaped.

Therefore the sine-shaped wave produces the minimum effective current in a condenser circuit.

Resonance.

When we have both inductance and capacity in the circuit the problem becomes of great practical importance, owing to the high pressures and large currents that are produced in distributing systems through the effects of resonance by comparatively low E.M.F.'s. Suppose that we have an

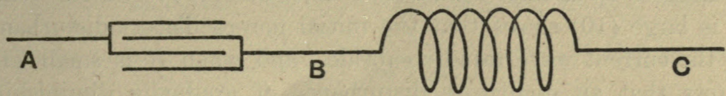

Fig. 21.

inductive coil (Fig. 21) in series with a condenser, and that an alternating P.D. is applied to the terminals A and C. Then in order to find the current we have to solve the equation

$$Ri + L\frac{di}{dt} + \frac{\int idt}{K} = E_1 \sin(\omega t + \alpha_1) + E_3 \sin(3\omega t + \alpha_3) + \dots \quad \dots(8).$$

The solution of this equation consists of two parts.

There is first the particular integral

$$i = \sum \frac{E_{2n+1} \sin(\overline{2n+1}\,\omega t + \alpha_{2n+1} - \beta_{2n+1})}{\sqrt{R^2 + \left\{L - \frac{1}{K(2n+1)^2\omega^2}\right\}^2 (2n+1)^2\omega^2}} \quad \dots(9),$$

where $$\tan \beta_{2n+1} = \frac{(2n+1)\,\omega\left\{L - \frac{1}{K(2n+1)^2\omega^2}\right\}}{R}.$$

The other part of the solution—the complementary function—is found by solving the equation

$$Ri + L\frac{di}{dt} + \frac{\int idt}{K} = 0.$$

When R^2 is greater than $\frac{4L}{K}$ the solution is of the form

$$i = A\epsilon^{-m_1 t} + B\epsilon^{-m_2 t} \quad \dots\dots\dots\dots\dots\dots(10),$$

where A and B are constants, and

$$m_1 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LK}}$$

and $$m_2 = \frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LK}}.$$

When $\frac{4L}{K}$ is greater than R^2 the solution is of the form

$$i = A\epsilon^{-\frac{Rt}{2L}} \cos\left\{\sqrt{\frac{1}{LK} - \frac{R^2}{4L^2}}\, t + \alpha\right\} \text{ ....(11)},$$

where A and α are constants. The complete solution of (8) is thus given by adding (9) and (10), or (9) and (11) together. When R is large (10) shows that the initial non-oscillatory disturbance of the current wave rapidly subsides, and when R is small, (11) shows that an oscillatory disturbance of gradually diminishing amplitude is superposed initially on the current. The period of oscillation of this disturbing effect is

$$\frac{2\pi}{\sqrt{\frac{1}{LK} - \frac{R^2}{4L^2}}} \text{ ....(12).}$$

If the applied P.D. wave were absolutely constant in shape, then, in practice, after a second or two (9) alone would give us the value of the current.

Now if $$LK(2n+1)^2\omega^2 = 1 \text{ ....(13)},$$
we see from (9) that $\beta_{2n+1} = 0$, and hence that the $(2n+1)$th harmonic of i is in phase with the $(2n+1)$th harmonic of the applied P.D. Its amplitude also is simply $\frac{E_{2n+1}}{R}$, and if R is small compared to $L\omega$, this term practically swamps all the other terms; thus the frequency of the alternating current is practically $\frac{(2n+1)\omega}{2\pi}$, and its effective value only slightly greater than $\frac{V_{2n+1}}{R}$. Also in this case the effective value of the P.D. across the choking coil will be nearly equal to

$$V_{2n+1}\left\{1 + \frac{(2n+1)^2\omega^2L^2}{R^2}\right\}^{\frac{1}{2}},$$

and the effective value of the P.D. across the condenser is approximately equal to

$$(2n+1)\,\omega L\frac{V_{2n+1}}{R} \quad\text{or}\quad \frac{V_{2n+1}}{(2n+1)\,\omega KR}.$$

It will be seen that if R be very small, or if the frequency $\frac{(2n+1)\omega}{2\pi}$ be very high, then these voltages may attain enormous values. The phase of the choking coil P.D. will be practically 90 degrees in advance, and that of the condenser P.D. 90 degrees behind the current. Hence the two P.D.'s are nearly in opposition to one another, and their resultant or the applied P.D. between A and C (Fig. 21) will be very small compared with either of them.

We see from (13) that the lowest value of the frequency at which resonance can occur is $\frac{1}{2\pi\sqrt{LK}}$; hence if the highest harmonic in the applied P.D. have a frequency less than this there will be no danger of resonance.

It is also to be noted that the above investigation shows that it is quite possible to obtain resonance effects in direct current circuits. The voltage curve of a direct current dynamo is never an exact straight line. The voltage has an alternating component due to the facts that the number of bars round the commutator is finite, and that the slots in the armature give rise to pulsations of the magnetic field. By suitably adjusting the value of the inductance of a choking coil put in series with a condenser between the mains, it is possible to get resonance, so that a large alternating current component starts in the circuit, and the pressure across the condenser attains a very high value. Duddell has shown that if we put a resonant circuit across a direct current arc formed between hard carbons, then resonance often ensues, and a large alternating current flows across the arc, causing it to emit a musical note. This happens even when accumulators are used instead of dynamos, and must be due to constantly recurring irregularities in the burning of the arc which continually renew oscillatory waves of the form given by (11).

In Fig. 22 we have supposed that the curve (I), which is parabolic, represents the shape of the wave of current in a circuit formed by a choking coil and condenser in series, and we have calculated the applied P.D. (E) necessary to produce this wave. It will be seen that it is a peaky wave very different from a sine curve. E_2 is the wave of P.D. at the terminals of the choking coil and is triangular in shape. E_1 gives the shape of the P.D. at the terminals of the condenser, and is very similar to a sine curve The diagram illustrates the general theorem that except when the applied P.D. wave is a sine curve, the wave of P.D. across the choking coil terminals is much more distorted from the sine shape than the wave of P.D. across the terminals of the condenser.

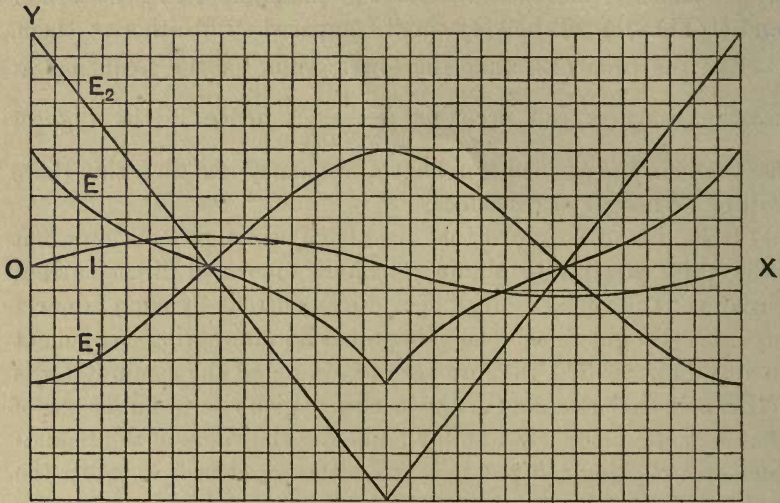

Fig. 22. Choking coil (E_1) and condenser (E_2) P.D.'s when the current wave (I) is part of a parabola.

Resonance of currents.

If we have (Fig. 23) a condenser with capacity K shunted by a choking coil of inductance L, then in certain cases the current in the main can be very small compared with that either in the choking coil or in the condenser. Let e, i_1

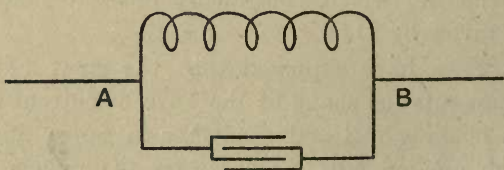

Fig. 23. Resonance of currents.

and i_2 be the instantaneous values of the P.D., the current in the condenser and the current in the choking coil respectively, then

$$i_1 = K\frac{de}{dt}; \quad e = L\frac{di_2}{dt}.$$

Now by equations (6) and (7) the effective values of the currents are

$$A_1 = \beta\omega KV \text{ and } A_2 = \frac{V}{\alpha\omega L},$$

where α and β are constants depending on the shape of e; these constants have their minimum value unity when the wave is sine-shaped. If i be the current in the main, then

$$i = i_1 + i_2,$$

and hence, noting that

$$i_1 = KL\frac{d^2 i_2}{dt^2},$$

we find that

$$\cos\phi = -\frac{\alpha}{\beta},$$

where ϕ is the phase difference (see Chapter VI.) between A_1 and A_2. If K and V are fixed, then the minimum value of the main current is $A_1 \sin\phi$, and in this case the choking coil current is $-A_1\cos\phi$.

$$\therefore\ LK(\alpha\omega)^2 = 1 \quad\text{.....................(14).}$$

If L and V are fixed, then the minimum value of the current in the main is $A_2 \sin\phi$, and it has this value when

$$LK(\beta\omega)^2 = 1 \quad\text{........................(15).}$$

Numerical examples.

1. For a sine wave, ϕ is 180 degrees, and the main current is zero when

$$LK\omega^2 = 1.$$

2. For a parabolic wave ϕ is 173° 46′, and if the condenser current is constant and equal to A_1, the minimum value of the current in the main is $0{\cdot}1086A_1$, and it has this value when

$$1{\cdot}001LK\omega^2 = 1.$$

Similarly if the current A_2 in the choking coil be kept constant, the minimum value of the main current is $0{\cdot}1086A_2$, and it has this value when

$$1{\cdot}013LK\omega^2 = 1.$$

3. For a triangular wave ϕ is 155° 54′. When the condenser current is constant and equal to A_1, then the minimum value of the main current is $0{\cdot}4083A_1$, and it has this value when

$$1{\cdot}013LK\omega^2 = 1.$$

When the condenser current varies and A_2 is constant, then the minimum value of the main current is $0{\cdot}4083 A_2$, and we have

$$1{\cdot}216 LK\omega^2 = 1.$$

If a condenser shunted by a choking coil be adjusted so that $LK\omega^2$ is unity, then $\frac{A_1}{A_2}$ equals $\alpha\beta$.

If A_1 were equal to A_2 the applied P.D. would be sine-shaped, and the greater this ratio the more distorted from the sine shape will be the wave of P.D. Also if A be the current in the main, the smaller A becomes compared to either A_1 or A_2 the nearer does the shape of the applied wave approximate to that of a sine curve. Hence we can use this theorem as a rough test for the shape of a wave of P.D.

Method of neutralising the inductive effect of a choking coil.

The inductive effect on a circuit of a coil (R, L) can be neutralised by shunting it as in Fig. 24 with a condenser K in series with a resistance R, provided that K equals $\frac{L}{R^2}$. Let e be the applied E.M.F. and let i_1, i_2 be the currents in the inductive coil and capacity respectively, also let

$$q = \int i_2 dt.$$

Then
$$e = Ri_1 + L\frac{di_1}{dt} \quad \text{.........................(1)},$$

and
$$e = \frac{q}{K} + Ri_2 = \frac{q}{K} + R\frac{dq}{dt} \quad \text{.....................(2)}$$

$$= \frac{1}{KR}\left\{Rq + L\frac{dq}{dt}\right\} \quad \text{............(3).}$$

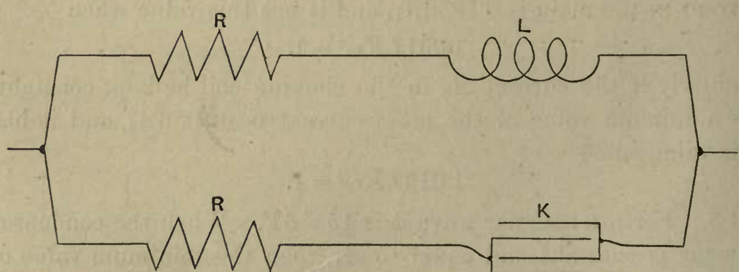

Fig. 24. Circuits in parallel which act exactly like a non-inductive resistance R when $L = KR^2$.

Subtracting (3) from (1), and solving the equation for $i_1 - \frac{q}{KR}$, we get

$$i_1 = \frac{q}{KR} + A\epsilon^{-\frac{R}{L}t}.$$

Therefore, ~~ultimately~~ [since i_1 and q are zero at the instant of closing the switch in the main, $A = 0$, and thus,] $i_1 = \frac{q}{KR}$.

Hence from (2) $e = R(i_1 + i_2)$.

But $i_1 + i_2$ is the current in the main, and therefore the combination acts like a non-inductive coil of resistance R.

If L be not equal to KR^2 the combination still acts like a non-inductive resistance R when the frequency is infinite. In this case the apparent resistance or impedance of the choking coil would be infinite, and the impedance of the condenser by itself would be zero.

Method of neutralising the capacity effect of a shunted condenser.

The capacity effect of a condenser K shunted by a resistance R can be neutralised by putting in series with it a purely inductive coil L shunted by a resistance R where L equals KR^2.

The equations in this case are

$$i = \frac{e_1}{R} + K\frac{de_1}{dt}$$

and

$$i = \frac{e_2}{R} + \frac{\int e_2 dt}{L} = K\frac{R}{L}e_2 + \frac{\int \frac{R}{L} e_2 dt}{R}.$$

Hence, proceeding as before,

$$e_1 = \int \frac{R}{L} e_2 dt + A\epsilon^{-\frac{t}{KR}}.$$

Thus, ~~ultimately~~ [since e_1 and $\int \frac{R}{L} e_2 dt$ are zero at the moment of closing the switch in the main, $A = 0$, and thus] $i = \frac{e_1 + e_2}{R}$.

Hence the combination acts exactly like a non-inductive resistance R.

Comparison of inductances by a voltmeter.

Suppose that we have two coils ab and cd with self inductances L and N respectively; let M be their mutual inductance. If we send an alternating current whose value is i through the coil ab, then the P.D. at its terminals will be given by

$$e = Ri + L\frac{di}{dt},$$

and an electrostatic voltmeter placed across ab will read V where

$$V^2 = R^2A^2 + L^2 \frac{1}{T}\int_0^T \left(\frac{di}{dt}\right)^2 dt$$
$$= R^2A^2 + L^2X^2.$$

Now if we connect b and c together, and put the voltmeter across a and d, then

$$e_1 = Ri + L\frac{di}{dt} - M\frac{di}{dt}.$$

Therefore $$V_1^2 = R^2A^2 + (L - M)^2 X^2.$$

Similarly if we join b and d and put the voltmeter across a and c, we should find V_2^2, where

$$V_2^2 = R^2A^2 + (L + M)^2 X^2.$$

Eliminating X^2 and R^2A^2 from these equations, we find that

$$\frac{M}{L} = 2\frac{V_1^2 + V_2^2 - 2V^2}{V_2^2 - V_1^2}.$$

Similarly we could find $\frac{M}{N}$ and hence $\frac{L}{N}$.

Provided the eddy currents are negligible, these equations are theoretically accurate. It is easily seen, however, that in certain cases a small error in reading the voltmeter may introduce a large error into the ratio of the inductances as calculated by the formula.

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CHAPTER IV.

Coefficients of self and mutual induction for electrostatic charges. Capacity of a conductor. Capacity between two conductors. Capacity of a condenser. Capacity of a concentric main. Ratio of electrostatic to electromagnetic units. The microfarad. The capacities of a triple concentric main. The capacity of a condenser formed by two long parallel cylinders. The capacity of a condenser formed by two long parallel cylinders, one wholly enclosed by the other. Condenser currents in concentric cables and in two parallel overhead wires. Two core cable. Three phase cables. Numerical example. Condenser currents in three phase working. Two phase cables with four separate conductors. Twin concentric cable. Model of a polyphase cable. Three core cable.

SUPPOSE that we have n conductors with potentials $v_1, v_2 \dots v_n$ respectively. Let there be electrical equilibrium when the potential of the first conductor is v_1 and all the other conductors are at zero potential. In this case the charge on the first conductor is proportional to v_1, and hence it may be written $K_{1.1} v_1$, where $K_{1.1}$ is a constant which is called the coefficient of self induction of the conductor for electrostatic charges. The charge on the second conductor will be negative and will be proportional to v_1. Let it equal $K_{2.1} v_1$ where $K_{2.1}$ is a constant and is negative. Similarly the charge on the nth conductor will be $K_{n.1} v_1$. Now consider another state of equilibrium, when the potentials of the conductors are $0, v_2, 0 \dots 0$. In this case the charge on the first conductor will be $K_{1.2} v_2$ and on the others $K_{2.2} v_2$, $K_{3.2} v_2 \dots K_{n.2} v_2$. Similarly we can write down expressions for the charges on the conductors when the pth is at a potential v_p and all the others are at zero potential. Now if we superpose all these systems, we get another system in a state of equilibrium, in which the charges on the conductors will be given by the linear equations

$$\left.\begin{aligned} q_1 &= K_{1.1} v_1 + K_{1.2} v_2 + K_{1.3} v_3 + \ldots\ldots + K_{1.n} v_n \\ q_2 &= K_{2.1} v_1 + K_{2.2} v_2 + K_{2.3} v_3 + \ldots\ldots + K_{2.n} v_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots(1).$$

It is obvious that $K_{1.1}$, $K_{1.2}$ ... depend only on the shapes and the relative positions of the conductors.

Consider two conductors R and S whose coefficients of self induction are $K_{r.r}$ and $K_{s.s}$ respectively. Suppose that all the other conductors are permanently connected to the earth. Put S to earth and charge R to the potential v_r. The work done during this process

$$= \int v_r dq_r,$$

but $q_r = K_{r.r} v_r$, and hence the work done is $\frac{1}{2} K_{r.r} v_r^2$. Now keep R at potential v_r and raise the potential of S to v_s. The work done by v_r in this stage

$$= \int v_r dq_r,$$

but $q_r = K_{r.r} v_r + K_{r.s} v_s$, and therefore $dq_r = K_{r.s} dv_s$. Hence, since v_r is constant, the work done is $K_{r.s} v_r v_s$. Similarly the work done by v_s during this process is $\frac{1}{2} K_{s.s} v_s^2$. Therefore the total work done in raising R and S to the potentials v_r and v_s is

$$\tfrac{1}{2} K_{r.r} v_r^2 + K_{r.s} v_r v_s + \tfrac{1}{2} K_{s.s} v_s^2.$$

Starting with the conductor S we find in the same way that the work is

$$\tfrac{1}{2} K_{r.r} v_r^2 + K_{s.r} v_r v_s + \tfrac{1}{2} K_{s.s} v_s^2.$$

Hence, equating these two expressions, we find that

$$K_{r.s} = K_{s.r}.$$

We may therefore call $K_{r.s}$ or $K_{s.r}$ the coefficient of mutual induction between R and S for electrostatic charges.

If we have n conductors, the total potential energy of the system is

$$\tfrac{1}{2} \Sigma K_{r.r} v_r^2 + \Sigma K_{r.s} v_r v_s.$$

From equations (1) we may write this by ordinary algebra in the form

$$\tfrac{1}{2} \Sigma q_r v_r.$$

Note the similarity of these expressions to the expressions for the electromagnetic energy of a system of currents,

$$\tfrac{1}{2}\Sigma L_{r.r}i_r^2 + \Sigma L_{r.s}i_r i_s$$

and

$$\tfrac{1}{2}\Sigma\phi_r i_r.$$

Again we may write the self energy of a single conductor in the form $\frac{1}{2}qV$. Now by Gauss's theorem $4\pi q = \Sigma RdS$, where we may suppose that dS is an element of the surface of the conductor itself. Also if ds be an element of the axis of the tube of force starting from dS,

$$V = \int Rds.$$

Therefore the self energy of the conductor

$$= \tfrac{1}{2}qV$$

$$= \frac{1}{8\pi}\Sigma RdS\int Rds.$$

But along ds, RdS is constant, therefore $RdS\int Rds = \int R^2 dSds$. Now $dSds = dv =$ the element of volume of a tube of force. Hence $RdS\int Rds = \int R^2 dv$, the integration being taken along the tube of force standing on dS. Therefore, integrating for the whole surface of the conductor, we see that

$$\Sigma\frac{R^2}{8\pi}dv$$

is an expression for the self energy of an electrified conductor, the integration being taken throughout all the space occupied by tubes of force.

Compare this with the expression

$$\Sigma\frac{H^2}{8\pi}dv$$

for the self energy of an electric current.

Capacity of a conductor.

The definition of capacity given on page 9 is a particular case of the following more general definition due to Maxwell.

The capacity of a conductor is its charge when its own potential is unity and that of all the other conductors is zero. In other words, it is the coefficient of self induction of the conductor for electrostatic charges. It is to be noted that any alteration in the position of any of the conductors generally alters the capacity of the conductor. In those practical cases, however, where we want to know this coefficient, all the conductors are fixed in position.

The capacity of a conductor is found by measuring the charge q which flows into it when it is connected to one terminal of an insulated battery whose electromotive force is v, the other terminal being connected to earth or to any of the surrounding conductors which are all earthed in this case as required by the definition. The ratio of q to v gives us the required capacity.

Electricians generally refer to the capacity of a conductor as the capacity between the conductor and all neighbouring conductors in parallel with the earth. In order to find the coefficients of mutual induction for electrostatic charges between the various conductors we find relations between the coefficients $K_{1.1}$, $K_{1.2}$, ... by measuring what is called the capacity between two of the conductors or between two groups of the conductors. The capacity between two conductors may be defined as follows.

Capacity between two conductors.

Let there be any number of conductors 1, 2, 3 ... n, and let 1 and 2 be insulated. Then, if, when all the conductors are initially uncharged a charge q be given to 1 and a charge $-q$ to 2, and if the potential of 1 now exceed that of 2 by $v_1 - v_2$, then the ratio of q to $v_1 - v_2$ is constant and is called the capacity between the two conductors. It is to be noticed that any alteration in the position of any of the conductors 3, 4, ... n generally alters the capacity between 1 and 2. Also connecting by fine wires any of the insulated conductors to one another or to the earth in general alters the capacity between 1 and 2, and so it is necessary to specify which of the conductors 3, 4, ... n are insulated from earth and whether any are joined together.

In practice the equal and opposite charges are given to the two conductors by connecting the terminals of an insulated battery to them, and the capacity between them is measured in exactly the same way as the capacity of an ordinary condenser.

Maxwell's equations on page 90 enable us to find in all cases an expression for the capacity between two conductors in terms of the coefficients $K_{1.1}$, $K_{1.2}$.... We shall apply them to find the capacity between two conductors, 1 and 2, when all other conductors in the neighbourhood are earthed. Give a charge q to 1 and a charge $-q$ to 2 and let their potentials be v_1 and v_2 respectively, then since v_3, v_4,... are all zero,

$$\left.\begin{aligned} q &= K_{1.1}v_1 + K_{1.2}v_2 \\ -q &= K_{2.1}v_1 + K_{2.2}v_2 \end{aligned}\right\} \qquad (2).$$

Solving these equations for v_1 and v_2 we find that

$$v_1 - v_2 = q\,\frac{K_{1.1} + K_{2.2} + 2K_{1.2}}{K_{1.1}K_{2.2} - K^2_{1.2}},$$

and

$$K = \frac{K_{1.1}K_{2.2} - K^2_{1.2}}{K_{1.1} + K_{2.2} + 2K_{1.2}} \qquad (3),$$

where K is the capacity between the two conductors. If $K_{1.1} = K_{2.2}$, then

$$K = \tfrac{1}{2}(K_{1.1} - K_{1.2}) \qquad (4).$$

An important practical case arises when the conductor 1 completely encloses the conductor 2. In this case when the charge on the conductor 2 is q, the induced charge on the inside of 1 will be $-q$. By definition, the difference of potential between 1 and 2 is the work done against the electric forces when we take a unit of positive electricity from 1 to 2. Hence the difference of potential between 1 and 2 depends only on the charge on the conductor 2, since the space inside 1 is completely screened from electrostatic induction from the outside. Hence, also, Maxwell's equation for q is

$$q = K_{2.1}v_1 + K_{2.2}v_2.$$

If v_1 equals v_2, q must be zero, therefore we must have $K_{2.2}$ equal to $-K_{2.1}$, hence

$$\frac{q}{v_2 - v_1} = K_{2.2} = -K_{2.1}.$$

Since this is true in all cases, it is true when the charge on the outside of the conductor 1 is zero, and hence it follows from our definition that the capacity between the conductors 1 and 2 is $K_{2.2}$ or $-K_{1.2}$.

If the conductor 2 be a metal sphere the centre of which coincides with the centre of a spherical cavity in the conductor 1, then if K be the capacity between 1 and 2,

$$K = K_{2.2} = -K_{1.2} = \frac{r_1 r_2}{r_1 - r_2},$$

where r_1 is the radius of the spherical cavity and r_2 is the radius of the metal sphere. When r_1 is infinitely great K equals r_2.

If the conductors instead of being separated from one another by air were separated by insulating materials like india-rubber, paper, oil, etc., then the capacities and coefficients of induction of the conductors would be altered. If they were embedded in a homogeneous insulating mass whose dielectric coefficient (specific inductive capacity) was λ, then the new constants would be $\lambda K_{1.1}$, $\lambda K_{1.2}$, etc. In electric lighting cables the conductors are as a rule separated by various materials whose dielectric coefficients are different. This considerably increases the difficulty of calculating the capacities between the various conductors, but as they are generally arranged in a symmetrical manner inside a metal sheath various useful formulae can be found giving all the capacities in terms of two or three constants. We will first however find an expression for the capacity of a condenser and calculate the capacities of concentric mains and of two parallel cylinders.

Capacity of a condenser.

A condenser consists of two equal insulated conductors 1 and 2, whose coefficients $K_{1.1}$, $K_{2.2}$, $K_{1.2}$ are very large compared with the mutual coefficients between 1 or 2 and the earth or other conductors. For a condenser we have, therefore, approximately, since $K_{1.1}$ is taken to be equal to $K_{2.2}$,

$$q = K_{1.1} v_1 + K_{1.2} v_2$$
$$-q = K_{1.2} v_1 + K_{1.1} v_2.$$

Now these equations must be true when v_2 is zero and thus $K_{1.1} = -K_{1.2}$ approximately. Using this relation, we at once find that the capacity between the two conductors, or briefly the capacity of the condenser, is approximately $K_{1.1}$ or $-K_{1.2}$.

In the theoretical condenser we suppose that $K_{1.2}$ is infinitely great compared with the other mutual coefficients, and hence $K_{1.1}$ or $-K_{1.2}$ is the capacity of the theoretical condenser.

Let a be the outer radius of the inner cylindrical conductor and b the inner radius of the outer conductor. Let also V_1 and V_2 be their potentials, $+q$ and $-q$ be their charges per unit length, and let k be the capacity per unit length, then

Capacity of a concentric main.

$$q = k(V_1 - V_2) \text{ ....}(5).$$

Now, from symmetry, the equipotential surfaces between the two cylinders are coaxial cylinders no matter how small a may be. Also the potential inside due to the charge on the outer cylinder is a constant and hence, if V be the potential at a point P distant x from the axis, where x lies between a and b, the force $\frac{dV}{dx}$ on unit of positive electricity placed at P will be the same, by Green's theorem (see page 8), as if the charge q were concentrated along an infinitely thin conductor coincident with the axis. Hence if dz be an element of the axis,

$$-\frac{dV}{dx} = \int_{-\infty}^{+\infty} \frac{qx\,dz}{(z^2+x^2)^{\frac{3}{2}}}$$

$$= \frac{2q}{x}.$$

This could also have been proved easily from Laplace's equation, which in this case gives us (page 6) $V = A + B \log x$; therefore (page 7)

$$\sigma = -\frac{1}{4\pi}\frac{dV}{dx} = -\frac{B}{4\pi a}$$

on the inner cylinder. Now $q = 2\pi a\sigma$ and hence $B = -2q$.

If λ be the dielectric coefficient of the insulating medium,

$$-\frac{dV}{dx} = \frac{2q}{\lambda x};$$

$$\therefore\ V_1 - V_2 = \frac{2q}{\lambda}\int_a^b \frac{dx}{x}$$

$$= \frac{2q}{\lambda}\log\frac{b}{a}.$$

Comparing this result with (5), we see that

$$k = \frac{\lambda}{2\log\frac{b}{a}},$$

or if K be the capacity, in electrostatic units, of a length l centimetres of concentric main, then

$$K = kl$$
$$= \frac{\lambda l}{2 \log_\epsilon \frac{b}{a}} \quad \text{.........................(6).}$$

Ratio of electrostatic to electromagnetic units. The microfarad.

The unit of capacity in either the electrostatic or the electromagnetic system is defined as the capacity of a condenser in which unit charge produces unit difference of potential between the conductors. Hence we may write

$$Q_e = K_e V_e \text{ and } Q_m = K_m V_m,$$

where the suffix e denotes that the quantities are measured in electrostatic units and the suffix m denotes that the quantities are measured in electromagnetic units. Now we can calculate by (6) the capacity K_e of a concentric main in electrostatic units. Also by charging it to a known P.D. through a ballistic galvanometer we can find its capacity in absolute electromagnetic units. The ratio of the electromagnetic to the electrostatic unit of capacity is theoretically equal to v^2 where v is the velocity of light. This quantity v has been shown experimentally by several observers to be 3×10^{10} centimetres per second, a number closely approximating to the observed speed of light. We may therefore write

$$K_m = \frac{1}{v^2} K_e.$$

The C.G.S. unit of electromagnetic capacity is too great for practical use. We therefore use two smaller units, the farad and the microfarad, the latter being the millionth part of the farad. In practice the microfarad is always used.

The farad is the capacity of a condenser which has a P.D. of one volt between its terminals when charged with one coulomb. Since the coulomb is the tenth of the C.G.S. unit of quantity, and the volt equals 10^8 C.G.S. units, we get

$$1 \text{ farad} = \frac{1}{10^9} \text{ C.G.S. unit,}$$

$$1 \text{ microfarad} = \frac{1}{10^{15}} \text{ C.G.S. unit.}$$

Hence if K be the capacity of a condenser in microfarads, and K_e be its capacity in electrostatic units,

$$K = \frac{10^{15}}{v^2} K_e$$

$$= \frac{1}{900000} K_e.$$

For example, the capacity of a concentric main of length l miles (160,900 l centimetres) in microfarads is given by

$$K = 0{\cdot}0776 \frac{\lambda l}{2 \log \frac{b}{a}} \quad \text{.........................(7)},$$

where the logarithm is to the base 10.

If we assume that the magnetic permeability and the dielectric coefficient have no dimensions, then the dimensions of the ratio of the electromagnetic to the electrostatic unit of capacity become (velocity)². By Coulomb's Law, assuming that the dielectric coefficient has no dimensions, we may write

$$\text{Force} \propto \frac{Q_e^2}{(\text{distance})^2};$$

thus $$Q_e \propto \text{space} \times \sqrt{\text{force}}.$$

Also $$Q_e \,.\, V_e \propto \text{work} \propto \text{force} \times \text{space};$$

therefore $$V_e \propto \sqrt{\text{force}}.$$

Similarly, assuming that the magnetic permeability has no dimensions, since the repulsion between two wires carrying equal currents i in opposite directions varies as i^2 (page 63),

$$i \propto \sqrt{\text{force}};$$

and $$Q_m \propto it \propto \sqrt{\text{force}} \times \text{time}.$$

Also $$Q_m V_m \propto \text{work} \propto \text{force} \times \text{space};$$

therefore $$V_m \propto \sqrt{\text{force}} \times \text{velocity};$$

thus $$\frac{Q_e}{Q_m} \propto \frac{\text{space}}{\text{time}} \propto \text{velocity},$$

and $$\frac{V_e}{V_m} \propto \frac{1}{\text{velocity}}.$$

Hence $$\frac{K_m}{K_e} \propto \frac{Q_m}{V_m} \cdot \frac{V_e}{Q_e} \propto \frac{1}{(\text{velocity})^2}.$$

To reduce the value of a given capacity when measured in electrostatic units to its value when measured in electromagnetic units we have to divide the first value by v^2. The electromagnetic C.G.S. unit is therefore v^2 times larger than the electrostatic. From Maxwell's equations expressing the propagation of an electromagnetic disturbance through a uniform medium, it follows that the velocity of propagation would be v. The agreement between v and the velocity of light is strong evidence of the soundness of his theory that light is a series of electromagnetic waves.

The capacities of a triple concentric main.

Let a be the outside radius of the inner main, b and c the radii of the middle main, and d the inside radius of the outer main. Then, denoting the capacity between the inner and the middle main by $C_{1.2}$, etc. we get

$$\left.\begin{aligned} C_{1.2} &= \frac{\lambda l}{2\log_\epsilon \frac{b}{a}} \\ C_{2.3} &= \frac{\lambda l}{2\log_\epsilon \frac{d}{c}} \\ C_{1.3} &= \frac{\lambda l}{2\log_\epsilon \frac{bd}{ca}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

In the case of $C_{1.3}$ the middle main is insulated. The presence of the inner cylinder thus increases the capacity between the two outers.

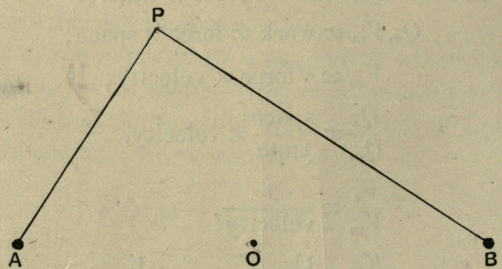

Fig. 25.

The capacity of a condenser formed by two long parallel cylinders.

Consider first the equipotential lines round two thin cylinders which are placed with their axes A and B (Fig. 25) perpendicular to the plane of the paper. Suppose that they are so thin that they may be regarded as lines, and that the cylinder A is charged with a quantity of electricity $+q$ per unit length and that the B cylinder has a charge $-q$. Now join AB and bisect it in O; let $AO=\frac{c}{2}$. Let v_A be the potential at any point P (Fig. 25) due to the action of the A wire alone and let v_A' be the potential at O. Then if the medium be air,

$$\frac{dv_A}{dr}=\frac{2q}{r}.$$

Thus
$$v_A - v_A' = \int_{r_1}^{\frac{c}{2}} \frac{2q}{r}\,dr$$
$$= 2q \log \frac{c}{2r_1}$$
where $AP=r_1$.

Similarly if v_B and v_B' be potentials at P and O due to the action of the wire B,

$$v_B - v_B' = -2q \log \frac{c}{2r_2}$$
where $BP=r_2$.

Now by the principle of superposition, if v be the potential at P due to both, then

$$v = v_A + v_B.$$

Also by symmetry the potential of O will be zero.

Therefore
$$0 = v_A' + v_B'.$$

Hence
$$v = 2q \log \frac{r_2}{r_1}.$$

Therefore the equation to the equipotential surface the potential of which is V is

$$V = 2q \log \frac{r_2}{r_1} \quad\text{..........................(1)},$$

or
$$\frac{r_1}{r_2} = \text{constant}.$$

Now by a well-known geometrical theorem the locus of a point P which moves so that the ratio of its distances from two fixed points A and B is constant, is a circle, and if C be its centre and r its radius,

$$CA \,.\, CB = r^2 \quad\text{..........................(2).}$$

These points are called inverse points with regard to the circle.

It will be seen that the equipotential surfaces are a series of cylinders surrounding A and B, all of which have A and B for inverse points. A particular case is the plane bisecting AB at right angles and passing through O. Now it follows from Green's theorem that, if we distribute electricity over one of the equipotential cylinders surrounding the wire A and if the surface density of the distribution at any point be $\frac{R}{4\pi}$, where R is the normal force at that point, then this distribution will be in equilibrium, the potential at external points will be unaltered, and the potential at all points inside this cylinder will be constant. Similarly we can replace the wire B by one of the equipotential cylinders surrounding it.

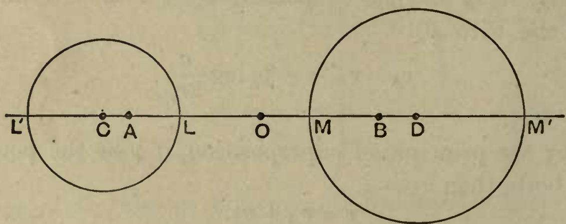

Fig. 26. A and B are the inverse points of the circles.

$CA \,.\, CB = CL^2$; $DA \,.\, DB = DM^2$.

$CL = a$, $DM = b$, $CD = d$, $AB = c$.

Suppose then that we have a solid cylinder LL' (Fig. 26) surrounding A and another MM' surrounding B. Let their potentials be V_1 and V_2 respectively, and their charges $+q$ and $-q$ per unit length as before, then from (1)

$$V_1 = 2q \log \frac{BL}{AL},$$

$$V_2 = 2q \log \frac{BM}{AM}.$$

$$\therefore\ V_1 - V_2 = 2q \log \frac{BL \,.\, AM}{AL \,.\, BM}$$

$$= \frac{q}{k},$$

where k is the capacity per unit length.

$$\therefore\ k = \frac{1}{2 \log \dfrac{BL}{AL} \cdot \dfrac{AM}{BM}} \quad \text{..................(2)}.$$

Now from Fig. 26,

$$\left.\begin{aligned} \frac{BL}{AL} &= \frac{BL'}{AL'} = \frac{2 \,.\, BC}{2 \,.\, CL} = \frac{BC}{CL} \\ \text{Also}\qquad \frac{AM}{BM} &= \frac{AM'}{BM'} = \frac{2 \,.\, AD}{2 \,.\, MD} = \frac{AD}{MD} \end{aligned}\right\} \quad \text{................(3)}.$$

Let the radii of the two cylinders be a and b respectively, let d be the distance between their centres, and let AB be c.

Then from (3),

$$\frac{BL \,.\, AM}{AL \,.\, BM} = \frac{BC \,.\, AD}{ab}$$

$$= \frac{b \,.\, CB}{a \,.\, DB},$$

since $\qquad DA \,.\, DB = b^2.$

We also have $\qquad CB\,(CB - c) = a^2,$

$$\therefore\ CB = \frac{c}{2} + \sqrt{a^2 + \frac{c^2}{4}}.$$

And $\qquad DB\,(DB + c) = b^2,$

$$\therefore\ DB = -\frac{c}{2} + \sqrt{b^2 + \frac{c^2}{4}}.$$

Also $\qquad d = CB + BD$

$$= \sqrt{a^2 + \frac{c^2}{4}} + \sqrt{b^2 + \frac{c^2}{4}},$$

$$\therefore\ c^2 d^2 = (d + a + b)(d + a - b)(d + b - a)(d - a - b) \text{......(4)},$$

an equation from which c can be rapidly found by the aid of logarithmic tables.

If λ be the dielectric coefficient of the medium and l the length of the cylinders, we get finally for the capacity K the formula

$$K = \frac{\lambda l}{2\left\{\log\left(\frac{c}{2a} + \sqrt{1+\frac{c^2}{4a^2}}\right) - \log\left(-\frac{c}{2b} + \sqrt{1+\frac{c^2}{4b^2}}\right)\right\}}$$

$$= \frac{\lambda l}{2\left\{\log_\epsilon \tan\left(90^\circ - \frac{\theta_1}{2}\right) - \log_\epsilon \tan\frac{\theta_2}{2}\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $\tan\theta_1 = \frac{2b}{c}$, and $\tan\theta_2 = \frac{2a}{c}$, and c is given by equation (4).

This equation may also be easily put in the form

$$K = \frac{\lambda l}{2\log_\epsilon\{\alpha + \sqrt{\alpha^2 - 1}\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where $$\alpha = \frac{d^2 - a^2 - b^2}{2ab}.$$

Formula (5) is however the most convenient one to use. If d be large compared with $a+b$, then approximately

$$K = \frac{\lambda l}{2\log_\epsilon\left\{\frac{d^2}{ab} - \frac{a^2+b^2}{ab}\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

When a equals b, the exact formula is,

$$K = \frac{\lambda l}{4\log_\epsilon\left\{\frac{d + \sqrt{d^2 - 4a^2}}{2a}\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

And finally, if d be large compared to a,

$$K = \frac{\lambda l}{4\log_\epsilon\left(\frac{d}{a} - \frac{a}{d}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

In formulae (5) to (9) if l be in miles and the logarithms are to the base 10, then to get K in microfarads we must multiply the right-hand side of the equations by 0·0776.

The cross section of the two cylinders is shown in Fig. 27. The inner one is supposed to have a charge $+q$ per unit length, and the outer one a charge $-q$. Let A and B be the inverse points common to the two circles, then, using the same notation as in the preceding example, we have

The capacity of a condenser formed by two long parallel cylinders one wholly enclosed by the other.

$$V_1 - V_2 = 2q \log \frac{b \,.\, CB}{a \,.\, DB}.$$

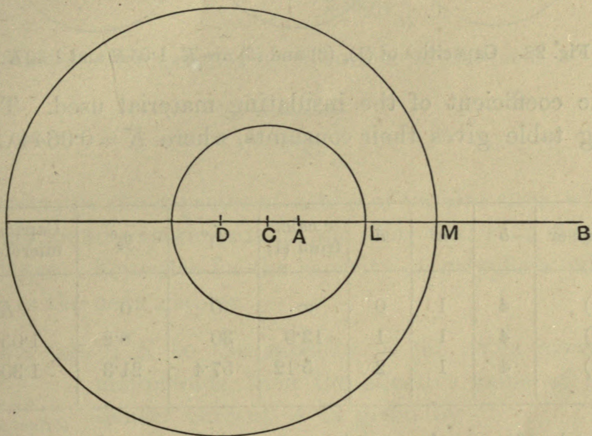

Fig. 27. A and B are the inverse points of the two circles.
$CA \,.\, CB = CL^2$; $DA \,.\, DB = DM^2$.
$CL = a$, $DM = b$, $CD = d$, $AB = c$.

Also
$$CB = \frac{c}{2} + \sqrt{a^2 + \frac{c^2}{4}},$$

$$DB = \frac{c}{2} + \sqrt{b^2 + \frac{c^2}{4}}.$$

$$\therefore d = DB - CB$$

$$= \sqrt{b^2 + \frac{c^2}{4}} - \sqrt{a^2 + \frac{c^2}{4}}.$$

$$\therefore c^2 d^2 = (a + b + d)(a + b - d)(b - a - d)(b - a + d) \ldots\ldots (1).$$

Hence
$$K = \frac{\lambda l}{2\left\{\log_\epsilon \tan \frac{\theta_1}{2} - \log_\epsilon \tan \frac{\theta_2}{2}\right\}} \ldots\ldots\ldots\ldots\ldots (2)$$

where $\tan \theta_1 = \frac{2b}{c}$, and $\tan \theta_2 = \frac{2a}{c}$, and c is found from (1).

Suppose that we have three cables whose sections are shown in Fig. 28 and suppose that they are l miles long and that λ is the

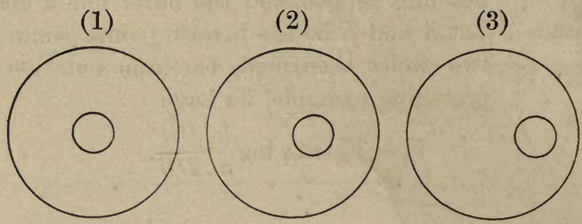

Fig. 28. Capacities of (1), (2) and (3) are K, $1{\cdot}05K$ and $1{\cdot}30K$.

dielectric coefficient of the insulating material used. Then the following table gives their constants, where $K = 0{\cdot}0644\lambda l$ microfarads.

Condenser	b	a	d	c calc. from (1)	$\theta_1{}^0$	$\theta_2{}^0$	Capacity in microfarads
(1)	4	1	0	∞	0	0	K
(2)	4	1	1	13·9	30	8·2	1·05 K
(3)	4	1	2	5·12	57·4	21·3	1·30 K

Formula (2) may be put into the form

$$K = \frac{\lambda l}{2 \log_\epsilon(\beta + \sqrt{\beta^2 - 1})} \quad \text{.................(3),}$$

where

$$\beta = \frac{a^2 + b^2 - d^2}{2ab}.$$

If d^2 be small compared with $b^2 - a^2$, then we have the approximate formula

$$K = \frac{\lambda l}{2 \log_\epsilon \left\{ \frac{b}{a} - \frac{b}{a} \frac{d^2}{b^2 - a^2} \right\}} \quad \text{.................(4).}$$

We see that the capacity is a minimum when the axis of the inner cylinder coincides with the axis of the outer one. The potential energy $\frac{Q^2}{2K}$ is therefore a maximum and the inner cylinder would be in unstable equilibrium if it were free to move in any direction.

If another metal cylinder be placed symmetrically inside the outer cylinder whose section is shown in (3) Fig. 28, then we should have a two core cable. In this case we can see from elementary principles that the capacity between the inner cylinder and the outer one will be increased by the presence of the new cylinder. Hence this capacity will be greater than 1·30 K, where

$$K = 0{\cdot}0776 \frac{\lambda l}{2 \log_{10} \frac{b}{a}}.$$

Manufacturers sometimes use formulae of the form

$$K = \frac{\alpha l}{\log_{10} \frac{b}{a}}$$

to predetermine the capacity of cables of various sizes, where α is a constant found experimentally, and they find such formulae of practical use. Formulae for the capacity of polyphase cables will be given in the next chapter.

Condenser currents in concentric cables and in two parallel overhead wires.

In this case if K be the capacity of the cable measured in microfarads, then the effective value of the condenser current A is given by

$$A = \alpha f K V 10^{-6} \text{ amperes},$$

where f is the frequency, V the effective voltage between the wires, and α a constant which has its minimum value 2π when the applied wave is sine-shaped. We have assumed that the potential drop due to the resistance of the cable is negligible.

Two core cable.

The two conductors are embedded in insulating material (Fig. 29) and are enclosed in a metal sheath which is connected to earth and is therefore at zero potential. Let $K_{1.1}$ be the capacity per mile of No. 1 conductor when No. 2 and the sheath are earthed, as defined on p. 91, let v_1 be its potential at any instant, and let $K_{1.2}$ be the coefficient of mutual induction per mile

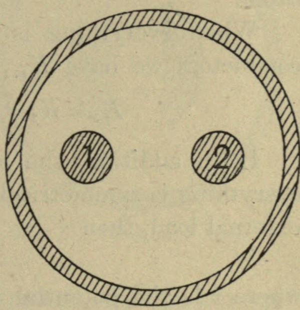

Fig. 29. Two core cable.

between the two conductors. We will assume that v_1 is constant throughout the whole length of the cable at any instant. Then if q_1 be the quantity of electricity on a mile of the No. 1 conductor,

$$q_1 = K_{1.1}v_1 + K_{1.2}v_2,$$

where v_2 is the instantaneous value of the potential of No. 2.

Similarly $$q_2 = K_{2.2}v_2 + K_{2.1}v_1.$$

These equations may be written:

$$q_1 = (K_{1.1} + K_{1.2})(v_1 - 0) - K_{1.2}(v_1 - v_2),$$
$$q_2 = (K_{2.2} + K_{1.2})(v_2 - 0) - K_{1.2}(v_2 - v_1).$$

Now if we have two small bodies 1 and 2 (Fig. 30) which are connected with each other and with an earthed conductor through condensers of capacities K_0, K_1 and K_2 as in the figure, and if the potentials of 1 and 2 are v_1 and v_2, then the charges on the conductors connected with 1 and 2 will be equal to q_1 and q_2 provided that $K_1 = K_{1.1} + K_{1.2}$, $K_0 = -K_{1.2}$ and $K_2 = K_{2.2} + K_{1.2}$.

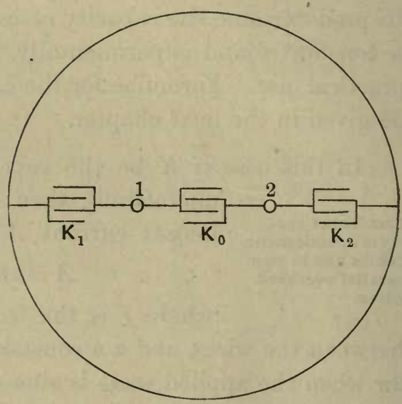

Fig. 30. Equivalent condensers.

We thus obtain an exact electrical model of a two core cable.

When everything is symmetrical with respect to the two conductors we have $K_{1.1} = K_{2.2}$, and thus

$$K_1 = K_2 = K_{1.1} + K_{1.2}, \quad K_0 = -K_{1.2}.$$

If, in addition, the alternator be insulated from earth and everything is symmetrical with regard to the middle point of the external load, then

$$v_1 = -v_2 = \tfrac{1}{2}v,$$

where v is the potential difference between the terminals of the machine. Now the condenser current for any main is defined as the rate of increase of charge upon the main, and hence, in this case, if $+i$ and $-i$ be the condenser currents for the two mains,

$$i=\frac{dq_1}{dt}=\tfrac{1}{2}(K_{1.1}-K_{1.2})\frac{dv}{dt}.$$

Leaving the symmetrical case for the moment, let No. 2 main be earthed at the alternator terminal, so that v_2 is zero. Then since v_1 equals v, the condenser currents are

$$i_1=K_{1.1}\frac{dv}{dt},\quad i_2=K_{1.2}\frac{dv}{dt}.$$

If v is the same in magnitude and wave form in the two cases, then at corresponding instants

$$\frac{i_1}{i}=\frac{2K_{1.1}}{K_{1.1}-K_{1.2}},\quad \frac{i_2}{i}=\frac{2K_{1.2}}{K_{1.1}-K_{1.2}}.$$

The currents have therefore constant ratios to one another, and if A, A_1 and A_2 be their effective values we have

$$A_1=\frac{2K_{1.1}}{K_{1.1}-K_{1.2}}A,\qquad A_2=\frac{-2K_{1.2}}{K_{1.1}-K_{1.2}}A.$$

Fig. 31 represents the section of a lead covered twin cable. The capacity between the two conductors is 0·345 microfarad per mile, and the capacity per mile between one conductor and the other in parallel with the sheathing is 0·53.

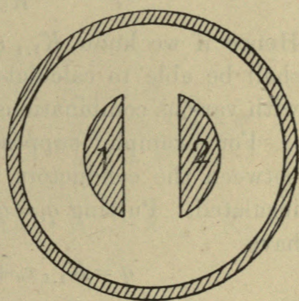

Fig. 31. Two core cable.

Hence $K_{1.1}=0·53$,

$\tfrac{1}{2}(K_{1.1}-K_{1.2})=0·345$,

$\therefore K_{1.2}=-0·16$.

Therefore if A be the condenser current when both mains are insulated, then when No. 2 main is earthed A_1 will equal $1·54A$ and A_2 will equal $0·46\,A$ and the current in the sheathing will be $(1·54-0·46)\,A$, *i.e.* $1·08\,A$.

We will suppose that three conductors are symmetrically embedded in a dielectric and surrounded by a metal sheath (Fig. 32). Then if v_0, v_1, v_2 and v_3 be the potentials of the sheath and of the three conductors respectively we have, with our usual notation,

Three phase cables.

$$q_0 = K_{0.0}v_0 + K_{0.1}v_1 + K_{0.2}v_2 + K_{0.3}v_3,$$
$$q_1 = K_{1.0}v_0 + K_{1.1}v_1 + K_{1.2}v_2 + K_{1.3}v_3,$$
$$q_2 = K_{2.0}v_0 + K_{2.1}v_1 + K_{2.2}v_2 + K_{2.3}v_3,$$
$$q_3 = K_{3.0}v_0 + K_{3.1}v_1 + K_{3.2}v_2 + K_{3.3}v_3.$$

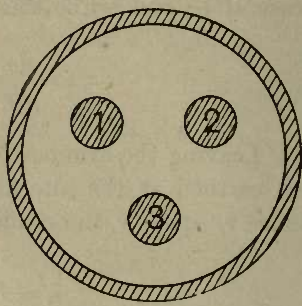

Fig. 32. Three core cable.

If we make $v_1 = v_2 = v_3 = v_0$, then there will be no charge on any of the internal conductors, since in practice they are completely screened by the sheath from electrostatic induction from the outside. Hence we obtain the three equations,

$$0 = K_{1.0} + K_{1.1} + K_{1.2} + K_{1.3},$$
$$0 = K_{2.0} + K_{2.1} + K_{2.2} + K_{2.3},$$
$$0 = K_{3.0} + K_{3.1} + K_{3.2} + K_{3.3}.$$

Now from symmetry,

$$K_{1.1} = K_{2.2} = K_{3.3}, \quad K_{1.2} = K_{2.3} = K_{3.1}, \quad K_{0.1} = K_{0.2} = K_{0.3}.$$

Using these values we reduce the three equations to the single equation

$$K_{0.1} + K_{1.1} + 2K_{1.2} = 0 \text{ ....} (a).$$

Hence if we know $K_{1.1}$ and $K_{1.2}$ we can find $K_{0.1}$, and then we shall be able to calculate the capacities which can be obtained with various combinations of the three conductors and the sheath.

For example, suppose that we wish to find the capacity between the conductors 1 and 2 when 3 and the sheath S are insulated. Putting $q_1 = q$, $q_2 = -q$ in the general equations, we have

$$q = K_{1.0}\,v_0 + K_{1.1}\,v_1 + K_{1.2}\,v_2 + K_{1.2}\,v_3,$$
$$-q = K_{1.0}\,v_0 + K_{1.2}\,v_1 + K_{1.1}\,v_2 + K_{1.2}\,v_3.$$

Hence $$2q = (K_{1.1} - K_{1.2})(v_1 - v_2),$$

and therefore by definition the capacity in question is

$$K = \tfrac{1}{2}(K_{1.1} - K_{1.2}).$$

We obviously get the same result when S and 3 are joined together by a fine wire or are put to earth.

Again, suppose we require the capacity between S and the conductor formed by joining 1 and 2, when 3 is insulated.

Give equal charges $\frac{1}{2}q$ to each of the mains 1 and 2. The induced charge on the inside of the sheath will be $-q$, and if there is no charge on the outside of the sheath this will be the total charge on S. Since the conductors 1 and 2 are practically screened by S, the capacity between them and S will be independent of the absolute value of the potential of S, and hence it will simplify our equations to put v_0 zero. Since v_1 equals v_2, the first and fourth equations now become

$$-q = 2K_{0.1}v_1 + K_{0.1}v_3,$$

and

$$0 = 2K_{1.2}v_1 + K_{1.1}v_3,$$

hence

$$q = -2v_1 K_{0.1}\left(1 - \frac{K_{1.2}}{K_{1.1}}\right).$$

Employing (a) we obtain for the capacity

$$K = 2\frac{(K_{1.1} + 2K_{1.2})(K_{1.1} - K_{1.2})}{K_{1.1}}.$$

The following is the complete list of the capacities that can be got from a three core cable.

(1) Capacity between 1 and 2

$$= \tfrac{1}{2}(K_{1.1} - K_{1.2}).$$

(2) Capacity between 1 and 2, 3

$$= \tfrac{2}{3}(K_{1.1} - K_{1.2}).$$

(3) Capacity between 1 and S (2 and 3 insulated)

$$= \frac{(K_{1.1} - K_{1.2})(K_{1.1} + 2K_{1.2})}{K_{1.1} + K_{1.2}}.$$

(4) Capacity between 1 and S, 2 (3 insulated)

$$= \frac{(K_{1.1} - K_{1.2})(K_{1.1} + K_{1.2})}{K_{1.1}}.$$

(5) Capacity between 1 and S, 2, 3

$$= K_{1.1}.$$

(6) Capacity between S and 1, 2 (3 insulated)

$$= 2\frac{(K_{1.1} - K_{1.2})(K_{1.1} + 2K_{1.2})}{K_{1.1}}.$$

(7) Capacity between 1, S and 2, 3

$$= 2(K_{1.1} + K_{1.2}).$$

(8) Capacity between S and 1, 2, 3

$$= 3(K_{1.1} + 2K_{1.2}).$$

If we measure (5) in the ordinary way, by reading the throw on a mirror galvanometer and comparing with the throw given by a standard condenser, we get $K_{1.1}$. A further measurement of (7) or (8) will give us a simple equation to find $K_{1.2}$.

Numerical example.

Let us take as an example the three phase 'clover leaf' extra high tension cable supplied to the Manchester Corporation by the British Insulated Wire Co.

The working pressure between the conductors is 6500 volts.

Working pressure between any conductor and the sheathing = 3750 volts.

Section of a conductor = 0·15 square inch = 0·97 sq. cm.

Minimum distance between conductor and sheathing } = 0·86 cm.
Minimum distance between any two conductors }

Insulating material, specially prepared paper. Mean dielectric coefficient $\lambda = 2{\cdot}8$.

By measurement, (7) was found to be 0·436 microfarad per mile, and (8) was 0·488 microfarad per mile.

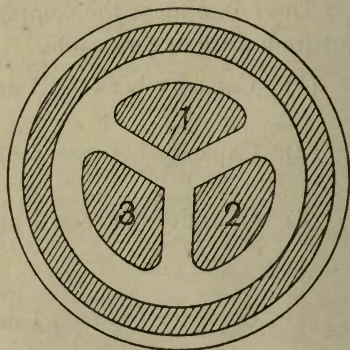

Fig. 33. 'Clover leaf' cable.

Therefore

$$\left.\begin{aligned} 2(K_{1.1} + K_{1.2}) &= 0{\cdot}436 \\ 3K_{1.1} + 6K_{1.2} &= 0{\cdot}488 \end{aligned}\right\};$$

hence $K_{1.1} = 0{\cdot}273$,

$$K_{1.2} = -0{\cdot}0553.$$

We deduce the other capacities by the formulae given above. The results are expressed in microfarads per mile.

(1) Capacity between 1 and 2 = 0·164.

(2) „ „ 1 and 2, 3 = 0·219.

(3) „ „ S and 1 = 0·245.

(4) Capacity between S, 1 and 2 $= 0{\cdot}262$.
(5) " " S, 1, 2 and 3 $= 0{\cdot}273$.
(6) " " S and 2, 3 $= 0{\cdot}391$.
(7) " " S, 1 and 2, 3 $= 0{\cdot}436$.
(8) " " S and 1, 2, 3 $= 0{\cdot}488$.

We also see that $K_{0.1}$, the coefficient of electrostatic induction between the sheath and a conductor, is $-0{\cdot}163$.

Condenser currents in three phase working.

In practical work v_0 is zero, and when the load is balanced (see Chapter XI.) we have

$$v_1 + v_2 + v_3 = 0.$$

In this case our equations become

$$q_1 = K_{1.1} v_1 + K_{1.2}(v_2 + v_3)$$
$$= K_{1.1} v_1 - K_{1.2} v_1$$
$$= (K_{1.1} - K_{1.2}) v_1.$$

Similarly $$q_2 = (K_{1.1} - K_{1.2}) v_2,$$
$$q_3 = (K_{1.1} - K_{1.2}) v_3.$$

Hence since $$\dot{i}_1 = \frac{dq_1}{dt},$$

we get $$\left.\begin{aligned} \dot{i}_1 &= 2K\frac{dv_1}{dt} \\ \dot{i}_2 &= 2K\frac{dv_2}{dt} \\ \dot{i}_3 &= 2K\frac{dv_3}{dt} \end{aligned}\right\},$$

where $K = \frac{1}{2}(K_{1.1} - K_{1.2})$ and $\dot{i}_1$, $\dot{i}_2$ and $\dot{i}_3$ are the capacity currents. Hence in calculating the capacity currents we can suppose that the conductors have no capacity, and are joined to the sheathing (Fig. 34) by three condensers each of capacity $2K$.

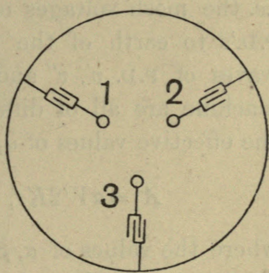

Fig. 34. Equivalent condensers.

For example, in the Manchester cable

described above, $2K$ is double the capacity between any two conductors, it therefore equals 0·328 microfarad per mile. If the working pressure between a conductor and the sheathing be 3750 volts, and the frequency be 50, then the minimum value of the condenser current in each conductor is $2\pi f 2KlV10^{-6}$, *i.e.* $0·386l$ ampere, where l is the length of the cable in miles.

The exact calculation of the capacity currents when the potential differences between the conductors and earth are not balanced is difficult, but a minimum limit can be fixed to the sum of the three condenser currents.

Let i_1, i_2 and i_3 be the three condenser currents, then

$$i_1 = K_{1.1}\frac{dv_1}{dt} + K_{1.2}\frac{dv_2}{dt} + K_{1.2}\frac{dv_3}{dt},$$

$$i_2 = K_{1.2}\frac{dv_1}{dt} + K_{1.1}\frac{dv_2}{dt} + K_{1.2}\frac{dv_3}{dt}.$$

Thus

$$i_1 - i_2 = (K_{1.1} - K_{1.2})\frac{d}{dt}(v_1 - v_2)$$

$$= 2K\frac{dv'}{dt} = a_1.$$

Similarly

$$i_2 - i_3 = 2K\frac{dv''}{dt} = a_2.$$

And

$$i_3 - i_1 = 2K\frac{dv'''}{dt} = a_3,$$

where v', v'' and v''' are the P.D.'s between adjacent conductors, *i.e.* the mesh voltages of the three phases. In general, when the P.D.'s to earth of the three conductors are out of balance, the waves of P.D. v', v'' and v''' between the three terminals of the machine are all of different shapes. Hence if A_1, A_2 and A_3 be the effective values of a_1, a_2 and a_3, we must write

$$A_1 = \alpha V' 2Kf, \quad A_2 = \beta V'' 2Kf, \quad A_3 = \gamma V''' 2Kf,$$

where the values of α, β or γ cannot be less than 2π, and f is the frequency. In these formulae K is in farads. Now we shall show in Chapter VIII. that the above equations prove that A_1, A_2 and A_3

can be represented by the sides of a triangle ABC. They also prove that I_1, I_2 and I_3, the effective values of i_1, i_2 and i_3, can be represented by lines OA, OB and OC, where O is a point which is not necessarily in the plane of the triangle ABC, but since $OA + OB + OC$ is never less than

$$\tfrac{1}{2}(AB + BC + CA),$$

therefore $I_1 + I_2 + I_3$ is never less than $\tfrac{1}{2}(A_1 + A_2 + A_3)$ and *a fortiori* it is never less than $2\pi(V' + V'' + V''')Kf$, where K is the capacity between any two conductors.

Two phase cables with four separate conductors.

Using the same notation as before, we have

$$q_1 = K_{1.1}v_1 + K_{1.2}v_2 + K_{1.3}v_3 + K_{1.4}v_4$$

when the sheath is earthed. Similarly we can write down the three other equations.

From symmetry (see Fig. 35)

$$K_{1.1} = K_{2.2} = K_{3.3} = K_{4.4}$$

$$K_{1.2} = K_{1.4} = K_{2.3} = K_{3.4}$$

$$K_{1.3} = K_{2.4}.$$

$$\therefore q_1 = K_{1.1}v_1 + K_{1.3}v_3 + K_{1.2}(v_2 + v_4).$$

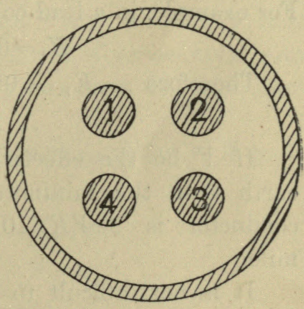

Fig. 35. Four core cable.

Now if the system is balanced (see Chapter XII.),

$$v_1 + v_3 = 0,$$

$$v_2 + v_4 = 0.$$

Hence

$$i_1 = \frac{dq_1}{dt}$$

$$= (K_{1.1} - K_{1.3})\frac{dv_1}{dt},$$

$$i_2 = (K_{1.1} - K_{1.3})\frac{dv_2}{dt},$$

$$i_3 = (K_{1.1} - K_{1.3})\frac{dv_3}{dt},$$

$$i_4 = (K_{1.1} - K_{1.3})\frac{dv_4}{dt}.$$

Therefore when we neglect the resistance of the conductors, the effect of capacity can be shown by imagining that the conductors have no capacity, but are joined by four condensers each of capacity $K_{1.1}-K_{1.3}$ connected star-wise between the conductors (Fig. 36). This capacity is double the capacity between two opposite conductors.

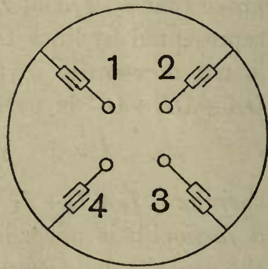

Fig. 36.
Equivalent condensers.

If we measure the capacity K_1 between 1 and 2, 3, 4, S joined in parallel, then

$$K_{1.1}=K_1.$$

Similarly, if we measure the capacity K_2 between 1, 3 and 2, 4, S, then

$$2(K_{1.1}+K_{1.3})=K_2.$$

Hence $$K_{1.3}=-(K_1-\tfrac{1}{2}K_2).$$

For example, in a lead-covered four core cable,

$$K_1=0{\cdot}234 \text{ and } K_2=0{\cdot}454.$$

Therefore $$K_{1.1}=0{\cdot}234 \text{ and } K_{1.3}=-0{\cdot}007.$$

$$\therefore K_{1.1}-K_{1.3}=0{\cdot}241.$$

If V be the effective pressure between any conductor and earth then the minimum value of the capacity current in a conductor is $2\pi VKf10^{-6}$, where K is 0·241 microfarad per mile.

It is not difficult to find expressions for all the capacities of the various condensers that can be made out of the four conductors and the sheath. For example, suppose we wish to find the capacity between 1 and 3 (Fig. 35).

We have on giving charges $+q$ and $-q$ to 1 and 3 respectively,

$$\left.\begin{aligned} q&=K_{0.1}v_0+K_{1.1}v_1+K_{1.2}v_2+K_{1.3}v_3+K_{1.2}v_4\\ \text{and}\quad -q&=K_{0.1}v_0+K_{1.3}v_1+K_{1.2}v_2+K_{1.1}v_3+K_{1.2}v_4\end{aligned}\right\}.$$

Thus $$2q=(K_{1.1}-K_{1.3})(v_1-v_3)$$

and therefore $$q=\tfrac{1}{2}(K_{1.1}-K_{1.3})(v_1-v_3).$$

Hence the capacity between a pair of opposite conductors is $\frac{1}{2}(K_{1.1}-K_{1.3})$, which is half the capacities of the condensers shown in Fig. 36.

In the twin concentric cable, a section of which is shown in Fig. 37, 1 and 4 are copper conductors and X is a third cylindrical copper conductor enclosing the other two and itself enclosed by a lead sheath. This cable is used for two phase working; 1 and 4 are what are ordinarily called the two outside conductors, and 2, 3 or X is their common return. The copper used in X is 1·414 times the copper used in either 1 or 4. When the system is balanced, the P.D. between 1 and X is equal to the P.D. between 4 and X as regards effective value but differs in phase from it by 90 degrees. The effective value of the P.D. between 1 and 4 is 1·414 times the effective value of the P.D. between 1 and X or between 4 and X. We also know that its phase difference from either of them is 135 degrees (Chapter XII.).

Twin concentric cable.

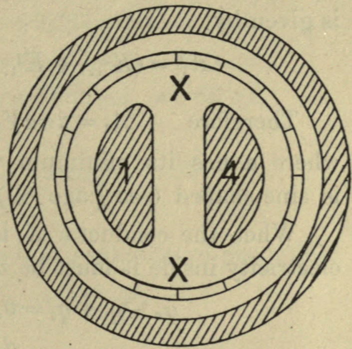

Fig. 37. Twin concentric cable.

Let v_1, v_x and v_4 be the potentials from earth of 1, X and 4 respectively, then as before

$$q_1 = K_{1.1} v_1 + K_{1.x} v_x + K_{1.4} v_4,$$

and $$q_4 = K_{4.1} v_1 + K_{4.x} v_x + K_{4.4} v_4.$$

From symmetry $K_{1.1} = K_{4.4}$ and $K_{1.x} = K_{4.x}$.

Since one conductor surrounds the other two, we must have q_1 and q_4 equal to zero when v_1, v_x and v_4 are all equal, therefore

$$K_{1.1} + K_{x.1} + K_{4.1} = 0.$$

Hence $$q_1 = K_{1.1}(v_1 - v_x) + K_{1.4}(v_4 - v_x),$$

and $$i_1 = K_{1.1} \frac{dv'}{dt} + K_{1.4} \frac{dv''}{dt}$$

where $$v' = v_1 - v_x \text{ and } v'' = v_4 - v_x.$$

If v' and v'' are similar waves, the effective values of which differ in phase by 90 degrees, then the effective values of the curves represented by $\frac{dv'}{dt}$ and $\frac{dv''}{dt}$ will also differ in phase by

90 degrees; if in addition the effective values of v' and v'' are equal, the effective value of the capacity current in either 1 or 4 is given by

$$A_1^2 = (K^2_{1.1} + K^2_{1.4}) \left\{ \text{mean value of } \left(\frac{dv'}{dt}\right)^2 \right\}.$$

Therefore $$A_1 = 2\pi\alpha V_{1.x} (K^2_{1.1} + K^2_{1.4})^{\frac{1}{2}} f,$$

where α has its minimum value unity when the wave of P.D. is sine-shaped (see page 80).

When the conductor X is earthed, since the total quantity of electricity inside it must be zero at every instant, we have

$$q_x + q_1 + q_4 = 0,$$

and $$i_x = \frac{dq_x}{dt},$$

therefore
$$\begin{aligned} i_x &= -(i_1 + i_4) \\ &= -(K_{1.1} + K_{1.4}) \left(\frac{dv'}{dt} + \frac{dv''}{dt}\right) \\ &= K_{1.x} \left(\frac{dv'}{dt} + \frac{dv''}{dt}\right). \end{aligned}$$

Thus
$$\begin{aligned} A_x &= -2\pi\alpha V_{1.x} \sqrt{2} K_{1.x} f \\ &= 2\pi\alpha V_{1.x} \sqrt{2} (K_{1.1} + K_{1.4}) f \\ &= 2\pi\alpha V_{1.4} (K_{1.1} + K_{1.4}) f. \end{aligned}$$

It follows from these formulae that A_x is always less than $\sqrt{2}A_1$.

As before we can show that

(1) The capacity between 1 and 4 $= \frac{1}{2}(K_{1.1} - K_{1.4})$.

(2) The capacity between 1 and X when 4 is insulated

$$= K_{1.1} - \frac{K^2_{1.4}}{K_{1.1}}.$$

(3) The capacity between 1 and X, 4 $= K_{1.1}$.

(4) The capacity between 1, 4 and $X = 2(K_{1.1} + K_{1.4})$.

The following are some of the data for a twin concentric cable made by the British Insulated Wire Co. for two phase work:

Working pressure between inner conductors = 2700 volts.

Working pressure between either inner and the outer conductor = 1900 volts.

Section of inner conductor = 0·025 sq. inch = 0·161 sq. cm.

Section of outer ring conductor = 1·414 × 0·161 sq. cm.
= 0·228 sq. cm.

Minimum distance between inner conductors = 0·56 cm.
" " between inner and outer = 0·63 cm.

The capacity between 1 and 4, X

$$= K_{1.1} = 0\cdot233 \text{ microfarad per mile.}$$

The capacity between 1, 4 and X

$$= 2\,(K_{1.1} + K_{1.4}) = 0\cdot370 \text{ microfarad per mile.}$$

Hence $K_{1.1} = 0\cdot233$ and $K_{1.4} = -0\cdot048$.

The capacity between 1 and 4

$$= \tfrac{1}{2}\,(K_{1.1} - K_{1.4}) = 0\cdot141 \text{ microfarad per mile.}$$

The capacity between 1 and X

$$= K_{1.1} - \frac{K^2_{1.4}}{K_{1.1}} = 0\cdot223 \text{ microfarad per mile.}$$

In calculating the above formulae for cables it is to be noted that we have supposed that the conductors are arranged symmetrically. This is the case in practice, and it is exceptional for example to find appreciable discrepancies in the values of the capacities between any two conductors of a three phase cable. When such discrepancies occur the above formulae have to be modified. The 'spiral,' that is, the twist of the cores round the central axis, in two and three core cables does not affect their symmetry. For cables with cores 0·1 of a square inch in section the spiral is in general one turn for about every eight feet of length. In cables made by some makers, however, the spiral is one turn for every four feet.

Model of a polyphase cable.

Let us consider the case of a cable with a metal sheath enclosing n conductors 1, 2, ..., n. If the diameter of the sheath be small compared with its length, as it always is in practice, then the inside conductors are practically screened from electrostatic induction from the outside, and so the coefficients of mutual induction between these conductors and outside bodies are quite negligible when compared with the mutual coefficients between any two conductors or between the conductors and the sheath. The equations for the charges on 1, 2, ... n are therefore

$$\left.\begin{array}{l} q_1 = K_{1.0}v_0 + K_{1.1}v_1 + \dots + K_{1.n}v_n \\ q_2 = K_{2.0}v_0 + K_{2.1}v_1 + \dots + K_{2.n}v_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots\dots\dots\dots(a).$$

When the potential of each of the n conductors is equal to that of the sheath,

$$q_1 = q_2 = \dots = q_n = 0$$

for all values of v_0. We thus obtain the n relations

$$\left.\begin{array}{l} K_{1.0} + K_{1.1} + \dots + K_{1.n} = 0 \\ K_{2.0} + K_{2.1} + \dots + K_{2.n} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots\dots\dots\dots(b).$$

Substituting in (a) the values of $K_{1.1}, K_{2.2}, \dots, K_{n.n}$ given by (b), we have

$$q_1 = -K_{1.0}(v_1 - v_0) - K_{1.2}(v_1 - v_2) - \dots - K_{1.n}(v_1 - v_n)$$
$$q_2 = -K_{2.0}(v_2 - v_0) - K_{2.1}(v_2 - v_1) - \dots - K_{2.n}(v_2 - v_n)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These equations show us that we may suppose the conductors to have no capacity, but that any one of them is connected to any of the others or the sheath by a condenser whose capacity equals the coefficient of electrostatic induction between the two with its sign changed. For example, the condenser connecting the conductor p to a conductor q will have a capacity equal to $-K_{p.q}$, and the condenser connecting it to the sheath will have a capacity $-K_{p.0}$.

Hence we can construct a model to illustrate the capacity effects of a polyphase cable as follows. Take n small conductors 1, 2, ..., n and join them to a conductor S by condensers of capacities $-K_{0.1}$, $-K_{0.2}$, ... Now join any two of them p and q by a condenser of capacity $-K_{p.q}$ and do this for every pair of conductors. The model thus constructed would act so far as capacity is concerned in a manner similar to the polyphase cable which has $K_{0.1}, K_{0.2}, \dots, K_{p.q}, \dots$ for its coefficients of mutual induction for electrostatic charges. The number of condensers required to construct the model would be

$$n + \tfrac{1}{2}n(n-1), \text{ that is, } \tfrac{1}{2}n(n+1).$$

Three core cable.

In the particular case of a three core cable, the capacities can be represented as in Fig. 38. The capacities of the condensers joining the conductors to the sheath will be $-K_{1.0}$, $-K_{2.0}$ and $-K_{3.0}$, and the capacities of the condensers joining the conductors will be $-K_{1.2}$, $-K_{2.3}$ and $-K_{3.1}$ respectively.

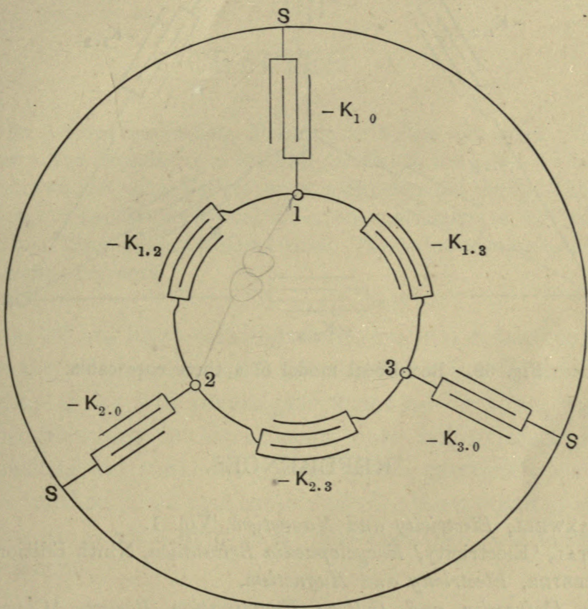

Fig. 38. Model of a three core cable.

When the three core cable is symmetrical,

$$K_{1.0} = K_{2.0} = K_{3.0}$$

and

$$K_{1.2} = K_{2.3} = K_{3.1}.$$

Hence when $K_{1.0}$ and $K_{1.2}$ are known, all the capacities of the cable can be found. It is an instructive exercise to find these capacities from Fig. 38. It will be found that they agree with the values which we have found for them earlier in the chapter. The capacities can also be represented as in Fig. 39.

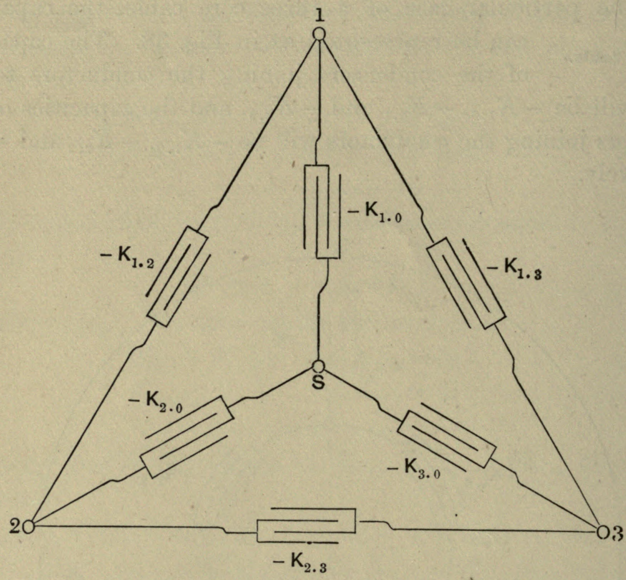

Fig. 39. Reciprocal model of a three core cable.

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CHAPTER V.

Formulae for a three core cable. Formula for a four core cable. Cable with n cores. The capacity of a cylinder parallel to the earth. The capacity between two horizontal parallel wires when near the earth. The capacity between two parallel horizontal wires one vertically over the other. The capacity of three phase overhead wires. The inductances of parallel wires with surface currents.

IN Chapter IV. we have considered the mutual relations between the capacities of the cores and the sheathing in polyphase cables. In this chapter we will investigate formulae for these capacities. These formulae are in some cases only approximate, but the approximations are sufficiently close to be practically useful, and the simple method employed, combined with the method of electrical images due to Lord Kelvin, is so powerful that it is deserving of attentive study.

Formulae for a three core cable.

We will suppose that the copper conductors or cores as we shall call them are three parallel cylinders, and we will first consider the case when each has a charge $+\frac{q}{3}$ per unit length. The equipotential surface whose potential is v is given by

$$v = C - 2\frac{q}{3}\log r_1 - 2\frac{q}{3}\log r_2 - 2\frac{q}{3}\log r_3,$$

where C is a constant and r_1, r_2 and r_3 are the distances of a point on the surface from the axes of the three cores respectively. If A be a constant this equation may be written in the form

$$r_1^2 r_2^2 r_3^2 = A^6.$$

Now, if the axes of the cores are at the angular points of an equilateral triangle whose centre is O, and if OP equals r, then it is easy to prove directly or by means of De Moivre's property of the circle (Loney's *Trigonometry*) that the last equation may be written

$$r^6 - 2a^3r^3 \cos 3\theta + a^6 = r_1^2 r_2^2 r_3^2 = A^6 \qquad \text{............(1)},$$

where θ is the angle which OP makes with a line passing through O and one of the angular points of the equilateral triangle, and a is the radius of the circle circumscribing it.

When the constant in (1) is zero, the curves are simply the three points where the axes of the cores cut the plane of the paper. When the constant is small, the equipotential curves are ovals which are nearly circular in shape and enclose the three points. When the constant equals a^6, the curves are given by

$$r^3 = 3a^3 \cos 3\theta,$$

which represents three loops, each enclosing a core. Each of these loops has a double point and two tangents at the origin, the angle between the tangents being 60 degrees.

When the constant is greater than a^6, we get a single curve enclosing the three cores and having three elevations and three depressions on it. For the curve passing through the point $r = 2a$, $\theta = 0$ the constant equals $(7a^3)^2$, and the maximum value of r for this curve is $2a$ and the minimum value is $1{\cdot}82a$. Hence the radii of this curve differ from the radius of the circle $r = 1{\cdot}91a$ by less than five per cent.

Now, by Green's theorem, we can replace any conductor by another surrounding it provided that the surface of the outer conductor is an equipotential surface of the system of distribution. Suppose then that the three core cable has the section shown in Fig. 40.

If b is the minimum distance of a core from the centre of the cable, then the equation to the boundaries of the cross sections of the three cores is

$$r^6 - 2a^3r^3 \cos 3\theta + a^6 = (a^3 - b^3)^2 \qquad \text{..............(2)}.$$

This equation has equal roots when

$$\cos^2 3\theta = \frac{b^3(2a^3 - b^3)}{a^6}.$$

If θ_1 is the positive solution of this equation, $2\theta_1$ is the angular breadth of the core as seen from the centre of the cable, and $120° - 2\theta_1$ is the angular breadth of the space between the cores as seen from the centre.

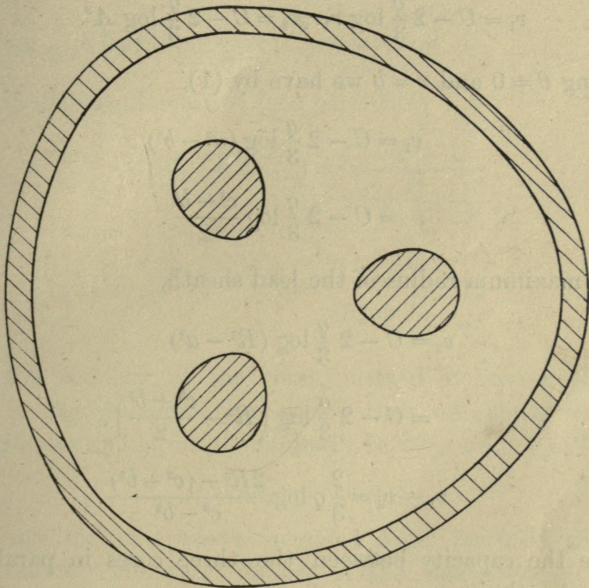

Fig. 40. Section of a three core cable, the equipotential surfaces being given by

$$r^6 - 2a^3r^3 \cos 3\theta + a^6 = \text{constant},$$

when the three cores are at the same potential.

If $b = \frac{1}{2}a$, as in Fig. 40, then $\theta_1 = 20°{\cdot}3$ nearly, and hence $2\theta_1 = 40°{\cdot}6$ and $120° - 2\theta_1 = 79°{\cdot}4$.

If c be the maximum distance of a point on a core from the centre of the cable, then from (2)

$$c^3 = 2a^3 - b^3;$$

and hence, if b is $\frac{1}{2}a$, c will be $1{\cdot}23a$.

This is the case illustrated in Fig. 40, the boundary of the lead sheath being supposed to coincide with the curve

$$r^6 - 2a^3r^3 \cos 3\theta + a^6 = (7a^3)^2,$$

so that its maxima and minima radii are $2a$ and $1{\cdot}82a$ respectively.

If v_1 be the potential of each core, and v_2 be the potential of the lead sheath, then

$$v_1 = C - 2\frac{q}{3}\log r_1 r_2 r_3 = C - 2\frac{q}{3}\log A^3.$$

Putting $\theta = 0$ and $r = b$ we have by (1),

$$v_1 = C - 2\frac{q}{3}\log(a^3 - b^3)$$

$$= C - 2\frac{q}{3}\log\frac{c^3 - b^3}{2}.$$

If R is a maximum radius of the lead sheath,

$$v_2 = C - 2\frac{q}{3}\log(R^3 - a^3)$$

$$= C - 2\frac{q}{3}\log\left(R^3 - \frac{c^3 + b^3}{2}\right),$$

and $$v_1 - v_2 = \frac{2}{3}q\log\frac{2R^3 - (c^3 + b^3)}{c^3 - b^3}.$$

Therefore the capacity between the three cores in parallel and the lead sheath is

$$\frac{3\lambda l}{2\log\dfrac{2R^3 - (c^3 + b^3)}{c^3 - b^3}},$$

where l is the length of the conductor, λ the dielectric coefficient, R the maximum inner radius of the sheath and b and c are the minimum and maximum distances of points on the cores from the centre of the cable.

The formula may also be written in the form

$$\frac{\lambda l}{2\log\dfrac{R}{a} + \dfrac{2}{3}\log\dfrac{1 - \dfrac{a^3}{R^3}}{1 - \dfrac{b^3}{a^3}}},$$

since $$2a^3 = b^3 + c^3.$$

Hence when $\frac{b}{a}$ is greater than $\frac{a}{R}$, the capacity is less than that of a concentric main whose inner radius is a and outer radius R. When $\frac{b}{a}$ equals $\frac{a}{R}$ the capacity equals that of this concentric main, and when $\frac{b}{a}$ is less than $\frac{a}{R}$ it is greater than it.

With our usual notation (see page 107)

$$3(K_{1.1} + 2K_{1.2}) = \frac{3\lambda l}{2 \log \frac{R^3 - a^3}{a^3 - b^3}},$$

and

$$K_{1.1} + 2K_{1.2} = \frac{\lambda l}{2 \log \frac{R^3 - a^3}{a^3 - b^3}} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

If the sections of the cores, instead of having the shapes shown in Fig. 40, were true circles, then we should expect that the equipotential surfaces would still be very similar to the curves

$$r_1 r_2 r_3 = \text{constant},$$

and hence that the above formulae could be used as first approximations. The exact shapes of the equipotential curves in this case could be found by the well-known laboratory method of tracing out the equipotential lines on a circular sheet of tinfoil whose boundary was maintained at zero potential whilst three circular copper electrodes were pressed on it at symmetrical points and maintained at constant equal potentials by a suitable battery.

If, however, we make the supposition that the circular cross sections of the wires are small compared with the cross section of the sheath we can find by the method of images approximate formulae to give us the values of $K_{1.1}$ and $K_{1.2}$.

Let O (Fig. 40*) be the centre of the section of the sheath by a plane perpendicular to its axis. Give charges $+\frac{q}{3}$ to each of the conductors the sections of which we suppose to be almost coincident with the points A, B and C. Let OA, OB and OC be each equal to a and let the angles between them be each equal to 120°. Let A', B' and C' be the inverse points (page 99) of A, B and C

with respect to the circle formed by the section of the inner surface of the sheath. Then

$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = R^2$

where R is the inner radius of the sheath. Let charges $-\frac{q}{3}$ be given to fine wires passing through the points A', B' and C' and parallel to the three original wires. Then if r_1, r_2, r_3, r_1', r_2' and r_3' are the distances of A, B, C, A', B' and C' from a point P where the potential is v, we should have if the sheath were removed

$$v = C - 2\frac{q}{3}\log\frac{r_1 r_2 r_3}{r_1' r_2' r_3'} \ldots (2).$$

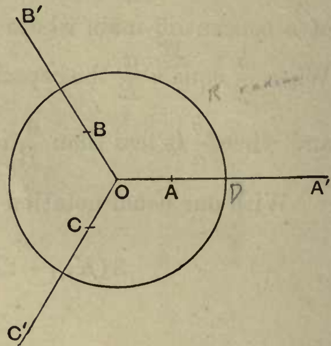

Fig. 40*. The images of the three wires A, B and C are at A', B' and C' where

$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = R^2$.

In finding formulae we replace the sheath by these images.

Now for all points on the circle, we have (page 101)

$$\frac{r_1}{r_1'} = \frac{r_2}{r_2'} = \frac{r_3}{r_3'} = \frac{a}{R} = \text{constant}.$$

Hence, substituting these values in (2), we see that the inner surface of the sheath is an equipotential surface of the six charged wires. Therefore, by Green's theorem, we can replace the three outside wires by the sheath without disturbing the equipotential surfaces inside. Conversely, as we desire in this case, we can replace the sheath by the three outside wires. It is necessary to suppose that the section of the wires inside is very small, otherwise the bounding surfaces of the three wires cannot be considered as equipotential surfaces which are determined approximately by (2).

When the sheath is at zero potential (2) becomes

$$v = 2\frac{q}{3}\log\frac{r_1' r_2' r_3'}{r_1 r_2 r_3} \cdot \frac{a^3}{R^3}.$$

Now if r be the radius of the circular cross section of a wire and v_1 be its potential, then, noting that (Fig. 40*)

$$AB = a\sqrt{3},\ AA' = \frac{R^2}{a} - a$$

and

$$AB'^2 = OB'^2 + OA^2 + OB' \cdot OA$$
$$= \frac{R^4}{a^2} + a^2 + R^2,$$

we get

$$v_1 = 2\frac{q}{3}\log\frac{\left(\frac{R^2}{a}-a\right)\left(\frac{R^4}{a^2}+a^2+R^2\right)}{r\,.\,a\sqrt{3}\,.\,a\sqrt{3}}\cdot\frac{a^3}{R^3}$$

$$= 2\frac{q}{3}\log\frac{R^6-a^6}{3R^3a^2r}.$$

Hence since the capacity between the three conductors in parallel and the sheath is $3(K_{1.1}+2K_{1.2})$ we get

$$K_{1.1}+2K_{1.2} = \frac{\lambda l}{2\log\frac{R^6-a^6}{3R^3a^2r}} \quad \text{...............(3).}$$

If we put $a-r$ for b in (1) and suppose that $\frac{r}{a}$ and $\left(\frac{a}{R}\right)^3$ are negligibly small, it is easy to see that the two formulae agree.

To find the capacity between the wires A and B we give a charge $+q$ to A and a charge $-q$ to B. We replace the sheath by wires A' and B' having charges $-q$ and $+q$ respectively. Assuming that there is no charge on C the equipotential surfaces are given by the equation

$$v = C - 2q\log\frac{r_1}{r_1'}\cdot\frac{r_2'}{r_2}.$$

Hence when the sheath is at zero potential,

$$v = 2q\log\frac{r_1'}{r_1}\cdot\frac{r_2}{r_2'}.$$

If v_1 be the potential of A and v_2 be the potential of B, then

$$v_1 = 2q\log\frac{\frac{R^2-a^2}{a}}{r}\cdot\frac{a\sqrt{3}}{\left(\frac{R^4+R^2a^2+a^4}{a^2}\right)^{\frac{1}{2}}}$$

$$= -v_2.$$

Hence since the capacity between the two wires is $\frac{1}{2}(K_{1.1}-K_{1.2})$ we get

$$\tfrac{1}{2}(K_{1.1}-K_{1.2}) = \frac{q}{2v_1},$$

and therefore

$$2(K_{1.1}-K_{1.2}) = \frac{\lambda l}{\log\frac{a\sqrt{3}}{r}\cdot\frac{R^2-a^2}{(R^4+R^2a^2+a^4)^{\frac{1}{2}}}} \quad \text{......(4).}$$

From equations (3) and (4) $K_{1.1}$ and $K_{1.2}$ can be readily determined.

By the help of the formulae given in Chapter IV. we can calculate all the capacities of a three core cable when the cores are of small section and not too close to one another in terms of these approximate values of $K_{1.1}$ and $K_{1.2}$.

Formula for a four core cable.

Assume that the equation to the boundaries of the sections of the four cores is

$$r^8 - 2a^4r^4 \cos 4\theta + a^8 = (a^4 - b^4)^2$$

where a is the distance of the axes of the cores from the centre of the cable, and b is the minimum distance of a conductor from the centre. Also assume that the equation to the boundary of the lead sheath is

$$r^8 - 2a^4r^4 \cos 4\theta + a^8 = (R^4 - a^4)^2$$

where R is the greatest value of the radius vector. Then proceeding in the same way as for a three core cable, we find that the equation to the equipotential surfaces is

$$v = C - 2\frac{q}{4}\log r_1r_2r_3r_4.$$

Hence if v_1 be the potential of the four cores and v_2 be the potential of the sheath

$$v_1 = C - 2\frac{q}{4}\log(a^4 - b^4),$$

$$v_2 = C - 2\frac{q}{4}\log(R^4 - a^4).$$

Thus
$$v_1 - v_2 = 2\frac{q}{4}\log\frac{R^4 - a^4}{a^4 - b^4}.$$

The capacity between the four cores in parallel and the sheath is therefore

$$\frac{2\lambda l}{\log\dfrac{R^4 - a^4}{a^4 - b^4}}.$$

If $R = ma$ and $b = \dfrac{a}{m}$, this becomes

$$\frac{\lambda l}{2\log m},$$

which is the same formula as for a concentric main whose outer radius is m times its inner one.

If c be the maximum distance of a point on the boundary of the cross section of a core from the centre of the cable, then

$$c^4 = 2a^4 - b^4.$$

And hence the formula becomes

$$\frac{2\lambda l}{\log \frac{2R^4 - (c^4 + b^4)}{c^4 - b^4}}.$$

When b and c are nearly equal to a, we can find approximate formulae for $K_{1.1}$, $K_{1.2}$ and $K_{1.3}$ by methods similar to that employed for the three core cable.

Cable with n cores.

Suppose that the n cores are all equal and parallel, and that they are arranged symmetrically in the sheath. If $\frac{q}{n}$ be the charge per unit length on each wire, then the potential at any point P inside the cylinder will be given by

$$v = C - 2\frac{q}{n}\log r_1 - 2\frac{q}{n}\log r_2 - \ldots\ldots - 2\frac{q}{n}\log r_n$$

$$= C - 2q \log (r_1 r_2 \ldots\ldots r_n)^{\frac{1}{n}}.$$

If the axes of the cores are arranged on a circle of radius a, this becomes

$$v = C - q \log \{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}\}^{\frac{1}{n}}$$

where θ is the angle which OP makes with OA, where A is one of the points of intersection of the axes of the conductors with a perpendicular plane.

The equation to the equipotential curve passing through the point $r = d$, $\theta = 0$ is

$$r^{2n} - 2a^n r^n \cos n\theta + a^{2n} = (d^n - a^n)^2.$$

This only meets the line $\theta = \frac{\pi}{n}$ when d is greater than $2^{\frac{1}{n}}a$. When d is very little greater than $2^{\frac{1}{n}}a$, the curve is rippled symmetrically and encloses the n cores. The maximum values of the radius vector are when $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}$ ......, and the minimum values when $\theta = \frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}$ .... The maximum values are each equal to d and the minimum values to $(d^n - 2a^n)^{\frac{1}{n}}$. For example, when n is 20 and d is $1{\cdot}2\,a$, the curve would differ from the circle $r = 1{\cdot}199\,a$ by less than one part in a thousand. Hence no great error is introduced by the assumption that the equipotential lines near the outer cylinder are circles.

When $d = 2^{\frac{1}{n}} a$ the equipotential lines are n loops, each enclosing a core and each having a double point at the origin. The angle between the tangents at the origin to one of these loops is $\frac{\pi}{n}$. Hence the length of the loops is much greater than their breadth.

When d is less than $2^{\frac{1}{n}} a$, there are n oval curves, one round each wire, and the smaller d is, the rounder these curves become. When d equals a, $\cos n\theta$ must equal unity, and therefore θ is $0, \frac{2\pi}{n}, \frac{4\pi}{n}$, ... that is, the curves are reduced to points which coincide with the axes of the cores. Hence, when d is nearly equal to a, the equipotential curves are n small rounded curves, and we may suppose the sections of the n cores to coincide with them.

Let v_1 be the potential of each core whose minimum distance from the centre of the cable is b, then

$$v_1 = C - 2\frac{q}{n}\log(r_1 r_2 \dots r_n)$$

$$= C - 2\frac{q}{n}\log(a^n - b^n).$$

Similarly, if v_2 be the potential of the sheath whose maximum inner radius is R, then

$$v_2 = C - 2\frac{q}{n}\log(R^n - a^n),$$

and

$$v_1 - v_2 = 2\frac{q}{n}\log\frac{R^n - a^n}{a^n - b^n}.$$

If c be the maximum distance of any point on a conductor from the centre of the cable, we have

$$c^n = 2a^n - b^n.$$

Hence the capacity between the n wires in parallel and the sheath is

$$\frac{n\lambda l}{2 \log \dfrac{2R^n - (c^n + b^n)}{c^n - b^n}}.$$

This may also be written in the form

$$\frac{\lambda l}{2 \log \dfrac{R}{a} + \dfrac{2}{n} \log \dfrac{1 - \left(\dfrac{a}{R}\right)^n}{1 - \left(\dfrac{b}{a}\right)^n}}.$$

Now $\dfrac{b}{a}$ will in general be greater than $\dfrac{a}{R}$, and hence the capacity will be less than that of a concentric main whose inner and outer radii are a and R respectively. If $bR = a^2$, or if n is infinite, the capacity will be the same as that of this concentric main.

The general appearance of the equipotential surfaces for the case of eight cores can be understood from Fig. 117, Chapter XV. When b is greater than 1·5 times a, the equipotential surfaces are practically circular cylinders having the same axis as the cable.

In the case of overhead wires which, unlike the conductors in cables, are not screened from outside influences by an enclosing metallic screen, we have to take into account the effect of the earth. We will first give the exact solution of the capacity of a single cylindrical wire parallel to the earth, and then give various practical approximate solutions for the case of several wires in parallel.

The capacity of a cylinder parallel to the earth.

Suppose that we have two equal cylinders parallel to one another, one charged with a quantity q of electricity per unit of length, and the other charged with a quantity $-q$. Then, by pages 9 and 99, the equipotential surfaces are given by

$$v = 2q \log \frac{r_2}{r_1},$$

where r_1 and r_2 are the distances of a point P from the inverse points A and B of the two circular sections made by a plane cutting the cylinders at right angles. Now, at every point on the plane bisecting AB at right angles, r_1 equals r_2, and therefore v equals zero, so that this plane is an equipotential surface. Since the earth is at zero potential, we see by Green's theorem, that the equation

$$v = 2q \log \frac{r_2}{r_1}$$

gives the equipotential surfaces for a single cylinder and the earth.

If a be the radius of the cylinder, v_1 its potential, and h be the height of its axis above the earth, then

$$v_1 = 2q \log \frac{r_2}{r_1},$$

$$= 2q \log \frac{a}{OA},$$

where O is the centre of the cylinder.

But $$OA \,.\, OB = a^2,$$

and $$OA + OB = 2h,$$

thus $$OA = h - \sqrt{h^2 - a^2},$$

and $$v_1 = 2q \log \frac{a}{h - \sqrt{h^2 - a^2}},$$

therefore $$K_{1.1} = \frac{l}{2 \log \dfrac{h + \sqrt{h^2 - a^2}}{a}}.$$

The capacity between two horizontal parallel wires when near the earth.

We suppose that the wires are at the same height, and we shall first find the capacity between the two wires in parallel and the earth. Let the charge on each wire be $+\frac{q}{2}$ per unit length. It is easy to see by the method of images that the equipotential surfaces will be the same as those due to four parallel wires, the new wires being the images of the old wires and charged with negative electricity. Let h be the height of the wires above the earth, then the distance between a wire and its image will be $2h$, and the equipotential surfaces will be given by

$$v = C - 2\frac{q}{2}\log r_1 - 2\frac{q}{2}\log r_2 + 2\frac{q}{2}\log r_3 + 2\frac{q}{2}\log r_4.$$

It is easy to see that C is zero; hence

$$v = q\log\frac{r_3 r_4}{r_1 r_2}.$$

If d be the horizontal distance between the wires, and if a, the radius of each wire, be small compared with d and h, we get the following approximate equation to give v_1 the potential of the wires,

$$v_1 = q\log\frac{2h\sqrt{d^2+4h^2}}{ad}.$$

Now if $K_{1.1}$ be the coefficient of self induction for electrostatic charges of each wire and $K_{1.2}$ the coefficient of mutual induction, the capacity of the two wires in parallel is $2(K_{1.1}+K_{1.2})$, and thus

$$K_{1.1}+K_{1.2} = \frac{l}{2\log\frac{2h}{a}+\log\left(1+\frac{4h^2}{d^2}\right)}.$$

We shall now find the capacity between the two wires.

Suppose that the charge on one wire is $+q$ per unit length and that the charge on the other wire is $-q$; then, since the capacity between the two wires is $\frac{1}{2}(K_{1.1}-K_{1.2})$, we find by the method of images that

$$K_{1.1}-K_{1.2} = \frac{l}{2\log\frac{2h}{a}-\log\left(1+\frac{4h^2}{d^2}\right)}.$$

In deducing these equations we have supposed that d and h are large compared with a.

Solving the equations we find

$$K_{1.1} = \frac{2l\log\frac{2h}{a}}{4\left(\log\frac{2h}{a}\right)^2-\left\{\log\left(1+\frac{4h^2}{d^2}\right)\right\}^2}$$

and
$$K_{1.2} = \frac{-l \log\left(1 + \frac{4h^2}{d^2}\right)}{4\left(\log \frac{2h}{a}\right)^2 - \left\{\log\left(1 + \frac{4h^2}{d^2}\right)\right\}^2}.$$

Comparing this with the formula obtained for a single cylinder, we see that the presence of a neighbouring cylinder increases the value of $K_{1.1}$.

If we make h infinite in the above formulae, then

$$K_{1.1} = \frac{l}{4 \log \frac{d}{a}}$$

and
$$K_{1.2} = -\frac{l}{4 \log \frac{d}{a}}.$$

The capacity between the wires in this case is approximately

$$\tfrac{1}{2}(K_{1.1} - K_{1.2}) = K_{1.1}$$
$$= \frac{l}{4 \log \frac{d}{a}}.$$

The capacity between two parallel horizontal wires one vertically over the other.

Let d be the distance between the wires, and h the height of the lower wire above the ground. Let the radius of each wire be a, and suppose that it is small compared with either d or h. Then, if the charge on the lower wire be q per unit length and that on the upper wire $-q$ per unit length, the equipotential surfaces are given by

$$v = 2q \log \frac{r_1'}{r_1} - 2q \log \frac{r_2'}{r_2},$$

where r_1 and r_1' are the distances of a point on the surface from the axis of the lower wire and its image respectively, and r_2 and r_2' are its distances from the upper wire and its image. If v_1 and v_2 be the potentials of the lower and upper wires, then

$$v_1 = 2q \log \frac{2h}{a} - 2q \log \frac{d+2h}{d}$$
$$= 2q \log \frac{2hd}{a(d+2h)}.$$

Similarly

$$v_2 = 2q \log \frac{d+2h}{d} - 2q \log \frac{2(d+h)}{a}$$

$$= 2q \log \frac{a(d+2h)}{2d(d+h)}.$$

Thus $$v_1 - v_2 = 2q \log \frac{4hd^2(d+h)}{a^2(d+2h)^2}.$$

Therefore the capacity between the two wires is approximately

$$\frac{l}{4 \log \frac{d}{a} + 2 \log \left\{1 - \frac{d^2}{(d+2h)^2}\right\}}.$$

If d be small compared with $2h$, this may be written

$$\frac{l}{4 \log \frac{d}{a} - \frac{2d^2}{(d+2h)^2}}.$$

If the wires had been in the same horizontal plane at a height h above the ground, then the capacity would be

$$\frac{l}{4 \log \frac{d}{a} + 2 \log \left\{1 - \frac{d^2}{d^2+4h^2}\right\}}.$$

Now since $\frac{d^2}{(d+2h)^2}$ is less than $\frac{d^2}{d^2+4h^2}$, the capacity between the wires for a given distance between them is a little smaller when they are arranged one over the other than when they are placed side by side, provided that the height of the lower wire in the one case is the same as the height of the two wires in the other. If however the mean height is the same in both cases we see, by writing $h - \frac{1}{2}d$ instead of h, that the capacities for the two arrangements are practically equal when d is small compared with $2h$.

The capacity of three phase overhead wires.

Suppose that the three wires are parallel to, and equidistant from, one another, the plane through the axes of the lower two being parallel to the earth. We will find the capacity between the lower two and the top wire. Let the charge on each of the lower two—(1) and

(3)—be $+\frac{q}{2}$ per unit length, and on the top wire—(2)—be $-q$, then, taking images (Fig. 41), we find for the equation of the equipotential surfaces

$$v = 2\frac{q}{2}\log\frac{r_1'}{r_1} - 2q\log\frac{r_2'}{r_2} + 2\frac{q}{2}\log\frac{r_3'}{r_3}$$

$$= q\log\frac{r_1' r_3' r_2^2}{r_1 r_3 r_2'^2}.$$

Let d be the distance between the axes of the wires and h the height of the two lower wires above the ground. Let a be the radius of each wire, which is supposed to be small compared with d and h. Then if v_1 be the potential of the lower wires, and v_2 the potential of the upper one, we have approximately

$$v_1 = q\log\frac{2hd\,(4h^2+d^2)^{\frac{1}{2}}}{a\,(4h^2+d^2+2hd\sqrt{3})},$$

$$v_2 = q\log\frac{a^2\,(4h^2+d^2+2hd\sqrt{3})}{d^2\,(2h+d\sqrt{3})^2}.$$

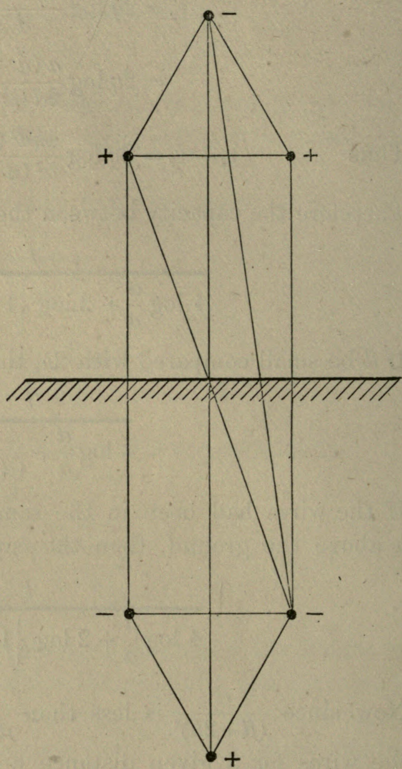

Fig. 41. Image of overhead three phase mains.

Thus $$v_1 - v_2 = q\log\frac{d^3 2h\,(4h^2+d^2)^{\frac{1}{2}}\,(2h+d\sqrt{3})^2}{a^3\,(4h^2+d^2+2hd\sqrt{3})^2}.$$

Therefore the capacity is

$$\frac{l}{3\log\frac{d}{a}+\log\frac{2h\,(4h^2+d^2)^{\frac{1}{2}}\,(2h+d\sqrt{3})^2}{(4h^2+d^2+2hd\sqrt{3})^2}}.$$

The expressions for the capacity between one of the lower wires and the other two, or between the three in parallel and the earth, can be written down similarly without much difficulty. They are however much more complicated.

In applying the above formulae it has to be remembered that we have made the assumption that the earth in the neighbourhood of the wires is a perfectly level plane of perfectly conducting matter. This assumption is in many cases not permissible. The presence of trees, rocks, buildings, etc. considerably complicates the problem; moreover the heights of the wires above the earth are generally not constant. In addition the wires are often 'spiralled' or 'barrelled' relatively to one another. For example if a, b and c be the insulators on the first pole and a', b' and c' be the corresponding insulators on the next, then No. 1 main would be connected between a and b', No. 2 main between b and c' and No. 3 main between c and a'. The mains would then probably be slung parallel to one another for a given number of poles and then another twist of 120° in the same direction would be given to them round the central axis. On long transmission lines one effect of this spiralling is to make the capacity and inductance between any two of the mains the same.

The capacity between an overhead wire and the earth is generally measured by charging the overhead wire to a given potential and then measuring its discharge to a good 'earth,' a water pipe for example, by means of the throw produced on a calibrated ballistic galvanometer. The ratio of the charge found in this manner to the potential is taken to be the capacity of the wire. Now after a prolonged drought the surface of the earth is a very bad conductor, and so it takes a considerable time to charge and discharge the condenser formed by the wire and the surface of the earth. Hence, as we should have expected from theoretical considerations, the capacity found in the above manner varies, sometimes by several hundred per cent., from day to day, depending on whether the ground is damp or dry. The value of the capacity of a main found by noting the charging current it takes when one pole of an alternator is connected to it and the other is put to earth corresponds practically with the capacity found by an instantaneous discharge of the main.

The inductances of parallel wires with surface currents.

The general equations of the electromagnetic field (see J. J. Thomson's *Elements of Electricity and Magnetism*, Chapter XI.) show that, when variable electromotive forces are applied to conductors, the resulting currents diffuse into the conductors from their surfaces in exactly the same way as heat would diffuse into the conductors if their surfaces were exposed to variable temperatures. There is this difference, however, that high electrical conductivity in the one problem corresponds to low thermal conductivity in the other (see Chapter XVI.). It follows also from the general equations that, when alternating E.M.F.'s of very high frequency act upon conductors, the currents are practically confined to very thin layers at the surfaces of the conductors, and it is easy to see that these surface currents must be distributed in such a way that they give rise to no magnetic force in the interior of the conductors.

The problem of finding the distribution of alternating currents in solid conductors can only be solved in the very simplest cases, when the frequency of the alternations and the electrical conductivity of the metal are finite. But, when the conductivity is supposed infinite, the problem of finding the distribution of the current can be solved by the aid of results already obtained in the solution of electrostatic problems.

When the conductivity is infinite, the currents are entirely confined to the surfaces of the conductors, just as the charges are in the case of an electrostatic distribution.

Now the magnetic force at a point at a distance r from an infinite straight filament of current of strength c is $2c/r$, the direction of the force being at right angles to the radius r in a plane at right angles to the filament. But, if the filament, instead of carrying the current c, is charged with q units of electricity per unit length, the electric force is $2q/r$, the direction of the force being along r. Hence, if we have a number of parallel cylindrical conductors of *any* cross-sections but of infinite length, and if the currents be confined to the surfaces of the conductors, the magnetic force is equal in magnitude to the electric force which the conductors would produce if they were charged with electricity in such a way that the surface density of the charge is equal to the surface density of the current; the magnetic force, however, is at right angles to the electric force. By the surface density of the current is meant the amount of current which flows, in a standard direction, per unit length across a line drawn on the surface of a conductor at right angles to its length; the surface density can therefore be positive or negative.

When any number of parallel cylindrical conductors are charged with electricity, the charges distribute themselves on the surfaces of the conductors in such a way that they produce no electric force in the interior of the conductors. Hence if currents flow along the conductors and have a surface density equal to that of the charges, the magnetic force will vanish at all points in the interior of the conductors. If the conductors have infinite conductivity, this will therefore be the distribution of current when alternating currents of proper magnitudes are made to flow along them.

In more precise language if i_1, i_2, ..., be the total currents in the cylindrical conductors, 1, 2, ... at any instant, these currents distribute themselves over the surface of the conductors in such a way that the surface density of the current is equal to that of the charge, when the conductors have the charges i_1, i_2, ..., per unit length. In addition, since the magnetic force at points outside the conductors is at right angles to the electric force, the lines of magnetic force are identical in form with the electric equipotential lines in planes perpendicular to the direction of the cylinders. Now the electric force R is equal to the magnetic force H. Hence it follows that the total flux of magnetic force which passes through a rectangle, which has two of its sides parallel to the direction of the cylinders and of unit length, is equal to the difference of electrostatic potential $v' - v''$ between these two sides. Therefore the total flux between two cylindrical conductors, per unit length in one case, is equal to the difference of electrostatic potential between them in the other case.

Since H equals R, we have

$$\Sigma \frac{H^2}{8\pi} dv = \Sigma \frac{R^2}{8\pi} dv,$$

so that (page 91) the magnetic energy in the one case is equal to the electrostatic energy in the other.

In order to fix our ideas, let us suppose that there are two sets of very long parallel conductors of the same length and that at one end both sets are connected with a single conducting plate cutting the conductors at right angles, while at the other end the two sets are connected with separate plates A and B. Now if, when the total current flowing from A to B through the system is I, the magnetic energy is $\frac{1}{2}LI^2$, the quantity L is called the self inductance of the system. When the conductivity of the conductors is not infinite, the distribution of current in each conductor at any instant depends not only upon I but also upon $\frac{dI}{dt}, \frac{d^2I}{dt^2}, \ldots$, and thus there is in this case no definite value of L. The problem becomes a little simpler when I is a simple harmonic function of the time, for then we can speak of the effective value of the self inductance. We can also calculate it, as Maxwell has done, for the simple case considered on page 46. In two cases only can we obtain definite values of L. The first case, in which the currents are constant and the conductivity is finite, has been sufficiently considered in pp. 52—62. The second case is that in which either the conductivity is infinite or the frequency is infinite. It is simpler to consider that the conductivity is infinite and we shall proceed on this assumption.

Now the E.M.F. between the ends of all the conductors of one set is the same and in the same direction, and hence, since their conductivity is infinite, the total flux of force passing between any two conductors of the same set is constant. If initially, before the E.M.F. was applied between A and B, there were no currents in the conductors, this flux of force is always zero. Hence, by what we have shown above, all the conductors of one set in the electrostatic problem are at the same potential.

Since the total flux between any two conductors of the same set is zero, the total flux between any two conductors of opposite sets is the same, and hence also the total flux of induction which is linked with any conductor of either set is the same for all the conductors of that set. The total flux between two conductors of opposite sets is easily seen to be simply LI. For, by page 56, we have

$$LI^2 = \Sigma\phi i,$$

where i is the total current embraced by the tube of force in which the flux is ϕ, and the summation takes in all the tubes of force. Since there is no magnetic force inside a conductor, i is constant for every tube linked with that conductor, and thus we may write

$$LI^2 = \Phi_A \Sigma i_A + \Phi_B \Sigma i_B,$$

since $\Sigma\phi$ has a constant value, say Φ_A, for every conductor of one set and a constant value Φ_B for every conductor of the other set. But the total flux between two conductors of opposite sets is $\Phi_A + \Phi_B$, and since Σi_A equals I and Σi_B also equals I, it follows that the total flux between two conductors of opposite sets equals LI, which proves our assertion.

We thus see that when the conductivity is infinite the total flux of force between two conductors of opposite sets, when the total current is unity, is equal to the self inductance of the circuit; this result is only true in the special case of infinite conductivity, though language is often used which implies that it is true when the conductivity is finite. The result is not true when the current is distributed over the sections of the conductors, because in this case some tubes of current are embraced by more tubes of magnetic force than others. The calculation given on pp. 55 to 59 definitely takes account of this variation in the number of tubes of magnetic force embraced by the various current tubes.

We can now easily express L in terms of K, the capacity between the two sets of cylinders, the cylinders of each set being connected 'in parallel.' For the flux of force $\Phi_A + \Phi_B$ between two conductors of opposite sets per unit length is LI/l, where l is the length of either conductor. Hence in the electrostatic problem if v be the difference of potential between two conductors of opposite sets, then since v must equal $\Phi_A + \Phi_B$, we have

$$\frac{ql}{K} = v = \Phi_A + \Phi_B = \frac{LI}{l},$$

and therefore, since I equals q,

$$L = \frac{l^2}{K}.$$

We can now write down the value of L for a pair of equal and parallel circular cylinders by the aid of the formula (8) of page 102. Thus

$$L = 4l \log_\epsilon \left(\frac{d + \sqrt{d^2 - 4a^2}}{2a} \right)$$

or when d is large compared with a

$$L = 4l \log_\epsilon \left(\frac{d}{a} - \frac{a}{d}\right).$$

The inductance L of a circuit formed by two parallel circular cylinders, one wholly enclosed by the other, is given by

$$L = 2l \log_\epsilon (\beta + \sqrt{\beta^2 - 1})$$

where $$\beta = \frac{a^2 + b^2 - d^2}{2ab}.$$

The radii of the cylinders are a and b respectively, and d is the distance between their axes (page 104). When d^2 is small compared with $b^2 - a^2$ this becomes

$$L = 2l \log_\epsilon \left(\frac{b}{a} - \frac{b}{a} \cdot \frac{d^2}{b^2 - a^2}\right),$$

and when d is zero (see also page 54),

$$L = 2l \log_\epsilon \frac{b}{a}.$$

We can write down in a similar manner approximate formulae for the inductances of the circuits formed by connecting in various ways the cores and the sheath of a polyphase cable by the aid of the formulae for the capacities given at the beginning of this chapter. For example, when the frequency of the alternating current is very high, the inductance of the circuit formed by the three cores in parallel and the sheath of the three phase cable illustrated in Fig. 40 can be found by the formula

$$L = \frac{2}{3} l \log_\epsilon \frac{2R^3 - (c^3 + b^3)}{c^3 - b^3}.$$

We can find in a similar way the formulae for the inductances of parallel wire circuits suspended above the earth and carrying currents of very high frequency by the help of the formula,

$$L = \frac{l^2}{K}.$$

For example, if part of a circuit be formed by a wire of length l and radius a at a height h above the earth, and if the earth form the remainder of the circuit, the earth currents will be concentrated on the surface of the earth, and we shall have by page 132

$$L = 2l \log \frac{h + \sqrt{h^2 - a^2}}{a},$$

or, when a is small compared with h,

$$L = 2l \log \frac{2h}{a}.$$

The logarithms in this chapter are all to the base ϵ.

In proving the above formulae it has been assumed that the total current in a conductor of either set has the same strength at all points on that conductor. This implies that the rates of variation of the charges which appear on the surfaces of the conductors, due to the E.M.F. between the two sets, are negligible, that is to say that the condenser currents are negligible.

Now this is true when we are dealing with direct currents which have arrived at their steady values. It is also true with alternating currents when the wires are infinitely thin and so have zero capacity. In the general case it is not true. For example, on a single phase line for the transmission of electric power at high pressure, the effective values of the currents in the two conductors at the generating station may be as high as fifty amperes, although the ends of the two conductors are not joined at the distributing station. As we proceed along the line from the generating station to the distributing station, the effective value of the current continually diminishes from fifty amperes to zero. The calculation of the electromagnetic energy at a given instant in this case is difficult even although we make the assumption that the conductivity of the wires is infinite. If v be the instantaneous value of the potential difference between the mains the electrostatic energy $\frac{1}{2}Kv^2$ can however be found at once.

Let the current at the generating end of the line when the line is loaded be $i_0 + i$, where i_0 is the capacity component of the current and i is the current taken by the power station. Then the electromagnetic energy stored in the field round the transmission lines lies in value between $\frac{1}{2}L\,(i + i_0)^2$ and $\frac{1}{2}Li^2$, where L is the value of the inductance calculated by the formulae given above. If the load current be large compared with the charging current, then $\frac{1}{2}Li^2$ will be approximately equal to the total electromagnetic energy.

When the line is very long, or when the frequency is so high that the distance travelled by light and therefore also the distance travelled by the electric disturbance during one alternation of the applied potential difference is comparable to the length of the conductors, other and more complex phenomena arise. The formulae given in this chapter should not be used in these cases without investigating whether the assumptions on which they are founded are permissible.

We have also neglected the brush discharge from the wires. When the voltage is 20,000 or upwards, the brush discharges from the wires are similar to those got from ordinary frictional electrical machines. If a thin wire be connected with a main at a high potential, the air immediately around it appears to glow, and, if the end of the wire is pointed, a current of air comes from it. Pieces of cotton hung from the mains repel one another, and all the ordinary electrostatic phenomena are produced. It is found in practice that the current and power taken to maintain this brush discharge is appreciable, and for this reason very thin wires must not be used for the transmission of electric power at very high pressures.

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OLIVER HEAVISIDE, *Electrical Papers*, Vol. 1, p. 42, 'On the Electrostatic Capacity of Suspended Wires,' and also p. 101, 'The Inductances of Suspended Wires.'

H. A. ROWLAND, 'Electromagnetic Waves and Oscillations at the Surface of Conductors.' *American Journal of Mathematics*, Vol. 11, p. 373, 1889.

Sir W. THOMSON [Lord KELVIN], 'On Alternate Currents in Parallel Conductors of Homogeneous or Heterogeneous Substance.' *B.A. Report*, 1890, p. 732.

For diagrams of equipotential surfaces in special cases see the figures in Chapter XV. and the references given there to papers discussing these curves.

CHAPTER VI.

The power factor. When the power factor is unity, the volt and ampere waves are similar. The maximum value of the power factor is unity. Geometrical interpretation of the power factor. Definition of phase difference. Time lag. Numerical examples. Zero power factor. Watt E. M. F. and wattless E. M. F. Impedance. Reactance. Watt current and wattless current.

The power factor.

IF e represent the instantaneous value of the P.D. across a circuit and i the instantaneous value of the current in it, then ei gives the instantaneous value of the watts expended in the circuit. The mean value of ei over a whole period gives us the rate at which work is being done in the circuit, or the power (W) being expended in it. The value of ei is sometimes negative for a fraction of a period and hence its mean value can be very small. Now the reading V of the voltmeter gives us the R.M.S. value of e, and A the reading of the ammeter gives us the R.M.S. value of i, but VA will not in general give us the mean value of ei. It is found convenient in practice to call the ratio of W to VA the power factor of the circuit, and we shall show that the maximum possible value of the power factor is unity. It may therefore be denoted by $\cos\phi$, where ϕ is a certain auxiliary angle of great use in graphical calculations. In mathematical symbols the power factor is defined by the equation

$$\cos\phi = \frac{\frac{1}{T}\int_0^T ei\,dt}{\left\{\frac{1}{T}\int_0^T e^2 dt \cdot \frac{1}{T}\int_0^T i^2 dt\right\}^{\frac{1}{2}}} = \frac{\int_0^T ei\,dt}{\left\{\int_0^T e^2 dt \cdot \int_0^T i^2 dt\right\}^{\frac{1}{2}}} \quad \ldots(1).$$

To prove this we notice, from the meaning of the integral sign, that

When the power factor is unity the volt and ampere waves are similar.

$$\frac{1}{T}\int_0^T e^2 dt = \underset{n=\infty}{\text{L}} \frac{e_1^2 + e_2^2 + e_3^2 + \ldots + e_n^2}{n},$$

where $e_1, e_2, \ldots, e_n$, are equidistant ordinates of the P.D. wave.

Dividing up the current wave into the same number of ordinates, we see that, if the power factor equals unity, then from (1),

$$\frac{(e_1 i_1 + e_2 i_2 + \ldots + e_n i_n)^2}{(e_1^2 + e_2^2 + \ldots + e_n^2)(i_1^2 + i_2^2 + \ldots + i_n^2)} = 1;$$

thus $$(e_1 i_1 + e_2 i_2 + \ldots)^2 - (e_1^2 + e_2^2 + \ldots)(i_1^2 + i_2^2 + \ldots) = 0,$$

and $$(e_1 i_2 - e_2 i_1)^2 + (e_1 i_3 - e_3 i_1)^2 + \ldots = 0.$$

Now since the square of a number is always positive, every term on the left-hand side is positive; and since the sum of them is zero, every term must be zero.

Hence $$e_1 i_2 - e_2 i_1 = 0;\ e_1 i_3 - e_3 i_1 = 0;\ \text{etc.}$$

Thus $$\frac{e_1}{i_1} = \frac{e_2}{i_2} = \frac{e_3}{i_3} = \ldots = \frac{e_n}{i_n}.$$

Therefore, at every instant, the ratio of the volts to the amperes is constant, which proves the theorem.

The maximum value of the power factor is unity.

Again since $(e_1 i_2 - e_2 i_1)^2 + (e_1 i_3 - e_3 i_1)^2 + \ldots$ is always greater than zero except when each term equals zero, it easily follows by going through the above proof backwards, that the power factor is less than unity except in the very particular case when the current and P.D. waves are the same curve drawn on different scales. It is convenient to call these waves similar waves. Hence if the power factor of a circuit is unity, the volt and ampere waves are similar, vanishing at the same instant and attaining all their maximum and minimum values at the same instants.

We may show in a similar manner that the power factor can not be less than -1. When the power factor is -1, e is negative when i is positive and *vice versâ*, but, as before, the ratio e to i is constant. Hence in this case also the waves are similar waves. They are drawn however on opposite sides of the axis.

It will be seen that $\cos\phi$ can only be equal to its limiting values, $+1$ and -1, in very special cases. As the angle ϕ is mainly useful in the application of graphical methods to alternating current problems, it is convenient to make the limitation that ϕ lies between 0° and 180°.

If a simple circuit is absolutely non-inductive and has no capacity, then

$$e = Ri,$$

where R is the resistance of the circuit. Substituting this value of e in (1) we see that the power factor equals unity. This could be proved directly as follows.

We have $$e^2 = R^2 i^2,$$

thus $$V^2 = R^2 A^2,$$

and $$V = RA.$$

Also $$ei = Ri^2,$$

therefore the mean value of ei = mean value of Ri^2,

hence $$W = RA^2$$
$$= V.A,$$

and the power factor $$= \frac{W}{VA} = 1.$$

Geometrical interpretation of the power factor.

By defining the power factor as the cosine of a certain angle ϕ, we are able in many cases to give a geometrical interpretation to the quantities involved, and can easily prove many algebraical relations between them by the help of known theorems in algebra and trigonometry.

We will consider a simple case.

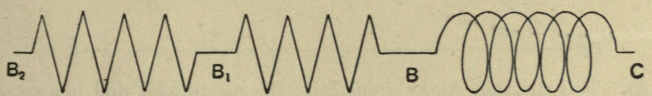

Fig. 42. Inductive resistance in series with non-inductive resistances.

Suppose that B_2C (Fig. 42) is part of an alternating current circuit. Let the resistances between BB_1 and B_1B_2 be R_1 and R_2 respectively, and suppose them non-inductive. Let e_1, e' be the instantaneous values of the P.D. between B_1 and C and between

B and C respectively, and let the instantaneous value of the current be i.

Then evidently we have always

$$e' = e_1 - R_1 i,$$

thus
$$e'^2 = e_1^2 + R_1^2 i^2 - 2R_1 e_1 i.$$

Hence by taking mean values for a whole period,

$$V'^2 = V_1^2 + R_1^2 A^2 - 2R_1 W \text{ ....(2)}$$

where V', V_1 and A are the effective values of the volts and amperes, and W is the mean value of $e_1 i$, that is, the power being expended in the circuit $B_1 C$. Denoting the power factor of this circuit by $\cos \phi_1$, we have

$$W = V_1 A \cos \phi_1,$$

and substituting in (2) we get

$$V'^2 = V_1^2 + R_1^2 A^2 - 2R_1 V_1 A \cos \phi_1.$$

If we now construct a triangle CBB_1 (Fig. 43) whose sides CB, BB_1 and $B_1 C$ are V', $R_1 A$, and V_1 respectively, we see by trigonometry that the angle CB_1B is ϕ_1. The cosine of CB_1B is the power factor of the circuit CB_1 (Fig. 42).

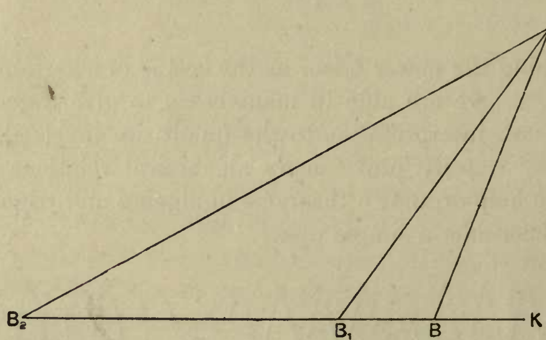

Fig. 43. This diagram shows graphically the magnitudes and phases of the potential differences in the circuit shown in Fig. 42.

Similarly producing BB_1 to B_2 (Fig. 43) and making B_1B_2 equal to $R_2 A$, it is easy to show that the cosine of the angle CB_2B is the power factor of the circuit CB_2 (Fig. 42) and that CB_2 (Fig. 43) is the effective value of the P.D. between C and B_2.

If V_2 denote this P.D. and ϕ_2 denote the angle CB_2B, then by trigonometry,

$$V_2 \sin \phi_2 = V_1 \sin \phi_1.$$

If we denote the angle CBK by ϕ', then

$$V_1 \cos \phi_1 - V' \cos \phi' = R_1 A,$$

thus

$$V_1 A \cos \phi_1 - V'A \cos \phi' = R_1 A^2.$$

Since $V_1 A \cos \phi_1$ is the work done between B_1 and C, it follows from this equation that $V'A \cos \phi'$ is the work done in the circuit BC (Fig. 42). Hence $\cos \phi'$ is the power factor of the circuit BC. Fig. 43, then, can be used to prove many relations between the volts, amperes and watts in the circuit. Thus to define the power factor as the cosine of an angle is a real help in understanding the relations of the various quantities involved.

Definition of phase difference.

It is convenient to call the angle whose cosine is the power factor the phase difference between the waves of P.D. and current. More generally, if e_1 and e_2 be two periodic functions of the same frequency, and ϕ be their phase difference, then

$$\cos \phi = \frac{\int_0^T e_1 e_2 dt}{\left\{\int_0^T e_1^2 dt \, . \int_0^T e_2^2 dt\right\}^{\frac{1}{2}}} \quad \text{..................(3),}$$

ϕ being an angle between 0 deg. and 180 deg.

The principal advantage of determining the value of this angle is to enable us to employ graphical methods.

For example, if e be the resultant of two P.D.'s e_1 and e_2, then at every instant

$$e = e_1 + e_2,$$

and

$$e^2 = e_1^2 + e_2^2 + 2e_1 e_2.$$

Hence, taking mean values over a whole period,

$$V^2 = V_1^2 + V_2^2 + 2V_1 V_2 \cos \phi \quad \text{..............(4),}$$

where ϕ is given by equation (3). We see then that V, the effective P.D. of the resultant, is the diagonal of the parallelogram constructed on V_1 and V_2 as adjacent sides when the angle between them is ϕ.

Time lag.

By the time lag of two periodic functions of the same frequency we mean the interval that elapses between the instants when they pass through their zero values in the positive direction. The angle of time lag may be defined as the angle described in an interval equal to the time lag by a uniformly rotating radius which makes one revolution in a time equal to the period of the alternating current.

If t_1, t_2 be the epochs at which e_1 and e_2 pass through zero in the positive direction, we can write

$$\omega t_1 = \alpha_1 \text{ and } \omega t_2 = \alpha_2.$$

The time lag between e_1 and e_2 is $t_1 - t_2$, and the angle of time lag is $\omega(t_1 - t_2)$, that is $\alpha_1 - \alpha_2$.

If $e_1 = E_1 \sin(\omega t - \alpha_1)$ and $e_2 = E_2 \sin(\omega t - \alpha_2)$, by substituting in (3) we find that

$$\cos\phi = \cos(\alpha_1 - \alpha_2),$$

therefore $$\pm\phi = \alpha_1 - \alpha_2.$$

In this case, then, the phase difference is numerically equal to the angle of time lag between e_1 and e_2, and, when e_1 and e_2 vanish at the same instant, then ϕ is zero or 180°. When e_1 and e_2 are not similar curves, then for no value of the time lag is the phase difference zero or 180°. This will be best understood by solving a few numerical examples.

Numerical examples.

In order to simplify the calculations we will suppose that one of the curves is a sine curve, and will find the phase difference between it and the curves of which the positive halves are shown in Fig. 44.

All the curves drawn give an effective voltage of 50, but the absolute value of the voltage has nothing to do with the phase difference, which depends only on the class of curve and its position relatively to the sine curve. If one of the curves represents a P.D. curve and the other the current curve to which it gives rise, then the cosine of the angle of phase difference between them will give the power factor.

The equations to the first halves of the curves shown in Fig. 44 are

(*a*) *Rectangle*, $e = V$,

(*b*) *Parabola*, $e = \frac{4\sqrt{30}}{T^2} V \left(\frac{T}{2} t - t^2\right)$,

(*c*) *Sine Curve*, $e = \sqrt{2} V \sin \frac{2\pi}{T} t$,

(*d*) *Triangle*, $e = 4\sqrt{3} V \frac{t}{T}$,

(*e*) *Inverted Parabolas*, $e = 16\sqrt{5} V \left(\frac{t}{T}\right)^2$,

(*f*) *Inverted Cubics*, $e = 64\sqrt{7} V \left(\frac{t}{T}\right)^3$.

All these curves have the same effective voltage V.

We shall find their phase differences with the curves

$$i = I \sin \frac{2\pi}{T} t \text{ and } i = I \sin \left(\frac{2\pi}{T} t - \alpha\right).$$

Curve	Maximum value of e	Height of C. G.	$\cos\phi$ with $I \sin \frac{2\pi}{T} t$	ϕ in degrees	$\cos\phi$ with $I \sin \left(\frac{2\pi}{T} t - \alpha\right)$
(*a*)	V	0·5 V	0·9003	25·8	0·9003 cos a
(*b*)	1·370 V	0·5476 V	0·9995	2·2	0·9995 cos a
(*c*)	1·414 V	0·5552 V	1·0000	0	cos a
(*d*)	1·732 V	0·5773 V	0·9928	6·75	0·9928 cos a
(*e*)	2·236 V	0·6708 V	0·9322	21·2	0·9322 cos a
(*f*)	2·646 V	0·7560 V	0·8628	30·4	0·8628 cos a

We might also have calculated the phase difference between any two of the curves shown in the figure. For example, the phase difference ϕ between the rectangle (*a*) and the very peaky curve (*f*) is 48·6 degrees, and the power factor ($\cos\phi$) is 0·6613.

We have proved at the beginning of this chapter that the power factor can only be unity, and consequently the phase difference can only be zero, when the ratio of e to i is constant throughout the whole wave. A first essential condition for zero phase difference is, thus, that e and i should both vanish at the same instant. This makes possible the second essential condition, namely, that the ratio of the ordinates of the two waves should be constant.

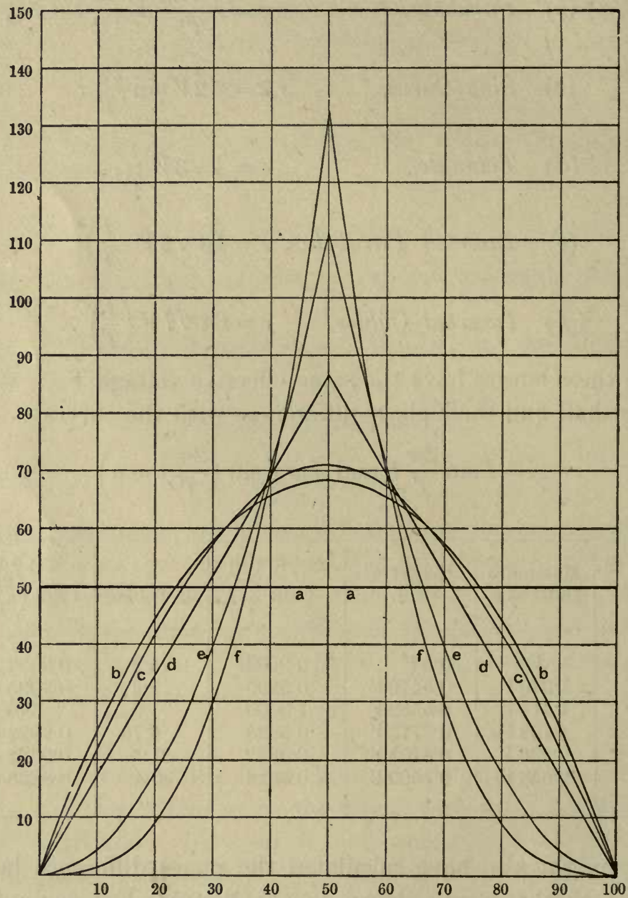

Fig. 44. Voltage curves, each of which has an effective value of 50. (*a*) Rectangle. (*b*) Parabola. (*c*) Sine curve. (*d*) Triangle. (*e*) Inverted parabolas. (*f*) Cubic curves.

Each of the curves shown in Fig. 44 is the first half of a symmetrical alternating curve. In such a curve the curve from $t=0$ to $\frac{1}{2}T$ is symmetrical about the ordinate corresponding to

$t = \frac{1}{4}T$, and the curve from $t = \frac{1}{2}T$ to T is the exact counterpart of the curve from $t = 0$ to $\frac{1}{2}T$, except that it lies on the opposite side of the axis. We may express these conditions by the equations

$$f(t) = f(\tfrac{1}{2}T - t) = -f(\tfrac{1}{2}T + t) \quad \text{.................(c).}$$

Since the curve is a continuous one it follows that $f(t) = 0$ when $t = 0$ and when $t = T$ and also when $t = \frac{1}{2}T$.

We can now show that, if ϕ_0 be the phase difference when the time lag is zero, and if α be the angle of lag between a symmetrical alternating curve and a sine curve, then

$$\cos\phi = \cos\phi_0 \cos\alpha \quad \text{.......................(5).}$$

To prove this, we have

$$\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi t}{T} - \alpha\right)dt = \int_0^{\frac{T}{2}} f(t)\sin\frac{2\pi t}{T}\,dt\cos\alpha$$

$$- \int_0^{\frac{T}{2}} f(t)\cos\frac{2\pi t}{T}\,dt\sin\alpha.$$

Putting $t = \dfrac{T}{2} - x$, we get, from (c),

$$\int_0^{\frac{T}{2}} f(t)\cos\frac{2\pi t}{T}\,dt = \int_{\frac{T}{2}}^{0} f\left(\frac{T}{2} - x\right)\cos\frac{2\pi x}{T}\,dx$$

$$= -\int_0^{\frac{T}{2}} f(t)\cos\frac{2\pi t}{T}\,dt.$$

The last integral therefore vanishes. If the limits in the last integral had been $\frac{1}{2}T$ to T, the integral would have vanished on account of the symmetry of $f(t)$ about the ordinate corresponding to $t = \frac{3}{4}T$. We thus obtain

$$\int_0^T f(t)\sin\left(\frac{2\pi t}{T} - \alpha\right)dt = \cos\alpha\int_0^T f(t)\sin\frac{2\pi t}{T}\,dt \text{......(d).}$$

Hence from (d) with the aid of (1) or (3)

$$\cos\phi = \cos\phi_0\cos\alpha.$$

It is instructive to give a graphical interpretation to this formula. Let *OAB* and *OBC* (Fig. 45) be two planes at right angles to one another, and let the angle *AOB* equal ϕ_0, and the angle *BOC* equal α, then the angle *AOC* will be ϕ.

Draw BC (Fig. 45) perpendicular to OC and join AC.

Then
$$OA^2 = OB^2 + AB^2$$
$$= OC^2 + CB^2 + AB^2$$
$$= OC^2 + AC^2.$$

Therefore the angle OCA is a right angle.

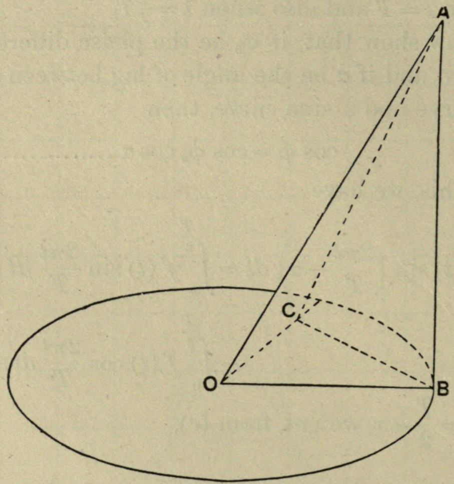

Fig. 45. Time lag and phase difference. For all positions of OC, the angle BOC gives the angle of time lag and the angle AOC gives the angle of phase difference between the two periodic quantities. The minimum value of the angle of phase difference is the angle AOB.

Now
$$OC = OA \cos AOC,$$
and
$$OC = OB \cos \alpha$$
$$= OA \cos \phi_0 \cos \alpha.$$
Thus
$$\cos AOC = \cos \phi_0 \cos \alpha\,;$$
therefore by (5) the angle AOC is the phase difference.

This gives us a graphical construction for the phase difference between a symmetrical wave and any sine wave. It is to be noted that this construction is in three dimensions. We shall return to this method of representing phase differences in Chapter VIII.

In practice it is the exception to have both curves symmetrical. Suppose that the current is a sine curve, and that there is no time lag between it and the P.D. curve. Then if we consider a family of P.D. curves all of the same height (see Fig. 20, Chap. III.), a little consideration will show that the power factor is a maximum when the peak of the wave occurs at the quarter period. Suppose now that the current is lagging, and that the P.D. wave has its maximum value before the quarter period. Then it is evident that the power factor will be less than if the P.D. wave were symmetrical. The maximum value of the power factor in this case occurs when the peak of the P.D. wave is in the second quarter.

The curves considered above have only one maximum ordinate during the half wave; they might however have several maxima and minima ordinates. Suppose for example the P.D. and current were given by the curves in Fig. 46. In this case the power factor would be 0·75 and the phase difference 41·4 degrees. It is fairly obvious that if the P.D. have a minimum value when the current has its maximum value, then the power factor will be low and the phase difference large even if there be no time lag between the current and the P.D.

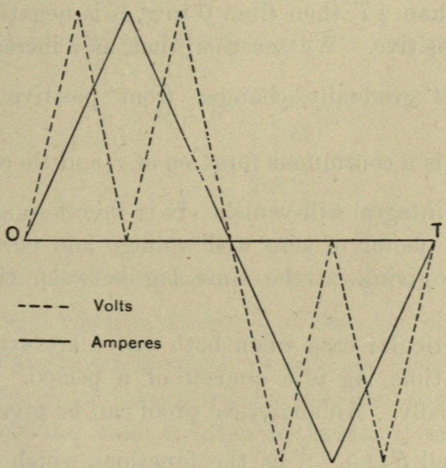

Fig. 46. Phase difference 41·4°. Power factor 0·75.

Zero power factor.

We shall now consider what happens when the power factor vanishes. We see from equation (1) that we have in this case

$$\int_0^T ei\,dt = 0.$$

Since in practice when the time is increased by $\frac{1}{2}T$ the values e_1 and i_1 become $-e_1$ and $-i_1$ respectively, we see that

$$\int_0^{\frac{T}{2}} ei\,dt = \int_{\frac{T}{2}}^{T} ei\,dt,$$

therefore $$\int_0^T ei\,dt = 2\int_0^{\frac{T}{2}} ei\,dt.$$

Let us suppose that the curve e only cuts the axis at points which are at distances $\frac{1}{2}T$ apart, and let us also make the same supposition about i.

When the time lag between e and i is zero, then ei is positive over the half period, and therefore the integral $\int_0^T ei\,dt$ is positive. When the time lag between e and i is $\frac{1}{2}T$, ei is negative over the half period and hence the integral is negative. When the time lag τ is less than $\frac{1}{2}T$, then from 0 to τ, ei is negative and from τ to $\frac{1}{2}T$, ei is positive. We see then that, as τ increases from zero to $\frac{1}{2}T$, $\int_0^T ei\,dt$ gradually changes from positive to negative. Hence $\int_0^T ei\,dt$ is a continuous function of τ, and therefore for some value of τ the integral will vanish. It is therefore always possible to get a power factor of zero with voltage and current waves of any shape by giving to the time lag between them a proper value.

In the particular case when both waves are symmetrical, the value of this time lag is a quarter of a period. This may be proved graphically. An analytical proof can be given as follows.

Let $f(t)$ and $F\left(t+\frac{T}{4}\right)$ be the functions which represent the voltage and current waves respectively. Then, since they are symmetrical, alternating curves, both $f(t)$ and $F(t)$ satisfy (c), and thus

$$\int_0^{\frac{T}{2}} f(t)\,F\left(t+\frac{T}{4}\right)dt = \int_0^{\frac{T}{2}} f\left(\frac{T}{2}-t\right)F\left(\frac{T}{4}-t\right)dt;$$

putting $t = T - x$, this
$$= -\int_{T}^{\frac{T}{2}} f\left(x - \frac{T}{2}\right) F(x - \tfrac{3}{4}T)\, dx$$
$$= \int_{T}^{\frac{T}{2}} f(x)\, F\left(x + \frac{T}{4}\right) dx$$
$$= -\int_{\frac{T}{2}}^{T} f(t)\, F\left(t + \frac{T}{4}\right) dt.$$

Thus
$$\int_0^T f(t)\, F\left(t + \frac{T}{4}\right) dt = 0.$$

This proves that, when both the current and voltage waves are symmetrical, the power factor vanishes when the time lag is a quarter of a period.

We have seen (page 78) that the charging current of a condenser is
$$i = K\frac{de}{dt},$$
where e is the potential difference at the condenser terminals. In this case
$$\int_0^T ei\,dt = K\int_0^T e\frac{de}{dt}\,dt$$
$$= \left[\frac{1}{2}Ke^2\right]_0^T$$
$$= 0,$$
since e has the same value after an interval equal to the period. We see also that when i is zero $\frac{de}{dt}$ is zero, and therefore e has a maximum or a minimum value. Assuming that e has only one maximum value in a period, we see that the time lag between e and i equals the time e takes to increase from zero to its maximum value.

Now e can have its maximum value at any time during the half period when it is positive. Therefore the time lag between e and i can have any value between 0 and $\frac{1}{2}T$. In all cases however the power factor is zero. This proves that we can infer nothing concerning the value of the time lag from a mere knowledge that the power factor is zero.

Watt E.M.F. and wattless E.M.F.

If W be the wattmeter reading in an alternating current circuit, V and A the voltmeter and ammeter readings respectively, and ϕ the angle of phase difference between the volt and ampere waves, a quantity which we have seen depends only on the shapes of these waves and their relative positions, then

$$W = VA \cos \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This follows from the definitions (1) and (3) above. Now if we suppose that V is resolved into two components whose values are $V \cos \phi$ and $V \sin \phi$ respectively, then these components are called the watt E.M.F. and the wattless E.M.F. respectively. We see from (6) that $V \cos \phi$ or the watt E.M.F. multiplied by the effective value of the current gives us the true mean power expended in the circuit.

If we have a simple inductive coil subjected to an alternating P.D., then

$$e = Ri + L\frac{di}{dt},$$

therefore $$VA \cos \phi = RA^2$$

and $$V^2 = R^2A^2 + \frac{L^2}{T}\int_0^T \left(\frac{di}{dt}\right)^2 dt.$$

Thus the effective value of Ri is $V \cos \phi$, and the effective value of $L\frac{di}{dt}$ is $V \sin \phi$. Hence we can suppose the applied P.D. split up into two components Ri and $L\frac{di}{dt}$, which have a phase difference of 90 degrees, and it is convenient to give names to the effective values of the two components. In the general case, however, when iron is present, we need to be careful when reasoning about the watt and wattless components of the E.M.F. as they do not seem to have much physical significance.

Impedance.

The impedance of a circuit is the ratio of the applied effective voltage to the effective value of the current produced. If V be the reading of a voltmeter placed across the circuit and A the reading of an ammeter in the circuit, then

$$Z = \frac{V}{A},$$

where Z denotes the impedance.

When direct current is used, Z is simply the resistance R of the circuit. With alternating currents, Z may be a very complicated function, as it depends on the shape and frequency of the applied potential difference wave, on eddy currents, the position of neighbouring circuits, capacity, inductance and magnetic permeability. In the case of a simple coil whose self inductance is constant, we have (page 43), when the potential difference wave is sine shaped,

$$Z^2 = R^2 + \omega^2 L^2.$$

If the wave be not sine shaped we can write (page 80)

$$Z^2 = R^2 + \alpha^2 \omega^2 L^2,$$

where α has its minimum value unity when the applied wave is sine shaped.

Reactance.

If V be the applied P.D., A the current and $\cos\phi$ the power factor of a circuit, then $\frac{V \sin\phi}{A}$ is called the reactance of the circuit. If there is no iron near the circuit and there are no eddy currents, then the reactance equals $\left\{\frac{V^2}{A^2} - R^2\right\}^{\frac{1}{2}}$ or $\alpha\omega L$, and if in addition the applied P.D. be sine shaped, then the reactance is simply ωL.

The reactance may also be defined as the ratio of the wattless P.D. to the current.

Watt current and wattless current.

Instead of supposing that the E.M.F. is resolved into two components, we may suppose that the current is so resolved. In this case $A\cos\phi$ is the watt current and $A\sin\phi$ is the wattless current.

For an inductive coil

$$e = Ri + L\frac{di}{dt}.$$

When e is of the form $E \sin \omega t$, then

$$i = \frac{R}{R^2 + \omega^2 L^2} E \sin \omega t - \frac{\omega L}{R^2 + \omega^2 L^2} E \cos \omega t.$$

The first term on the right-hand side, being in phase with e, may be considered as the watt component of the current, and the other term may be considered as the wattless component of the current.

In this case we have

$$\text{the watt current} = A \cos \phi = \frac{RV}{R^2 + \omega^2 L^2} = \frac{RA^2}{V},$$

$$\text{and the wattless current} = A \sin \phi = \frac{\omega L V}{R^2 + \omega^2 L^2} = \frac{\omega L A^2}{V}.$$

If e were the parabolic wave whose equation (b) is given above, then we can show that

$$A \cos \phi = \frac{V}{R} \left\{ 1 - 40 \left(\frac{Lf}{R} \right)^2 + 1920 \left(\frac{Lf}{R} \right)^4 - 7680 \left(\frac{Lf}{R} \right)^5 \frac{\epsilon^{\frac{R}{2Lf}} - 1}{\epsilon^{\frac{R}{2Lf}} + 1} \right\}$$

$$= \frac{RA^2}{V}.$$

This proves that in general $A \cos \phi$ is a very complicated function of the quantities involved, and reasoning based on it has to be closely examined.

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CHAPTER VII.

Argand's method of representing a complex variable. Vectors. Addition of vectors. Polygon law for compounding vectors. Multiplication of a vector by a complex number. Division of a vector by a complex number. Application to the theory of alternating currents. The currents in a divided circuit. Inductive coil in series with choking coil shunted by a non-inductive resistance. The apparent resistance and inductance of branched circuits. The currents in a branched circuit when mutual inductance is taken into account. Condition that the energy expended should have a stationary value. Important consequences in this case. Graphical solution. References.

IT is convenient in Algebra and Trigonometry to introduce the conception of the square root of negative unity, and to make the convention that it must obey all the ordinary algebraical laws. An expression of the form $x + y\sqrt{-1}$ is called a complex number, and has many properties which make it a great aid in calculation. The expression $x - y\sqrt{-1}$ is said to be conjugate to $x + y\sqrt{-1}$, and $\sqrt{x^2 + y^2}$, the square root of the product of the two, is called the modulus of either. The following three fundamental theorems are proved in text-books on algebra.

1. If a, b, c and d are real quantities and

$$a + b\sqrt{-1} = c + d\sqrt{-1},$$

then

$$a = c \quad \text{and} \quad b = d.$$

2. $$\frac{1}{a + b\sqrt{-1}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\sqrt{-1}.$$

3. $$\frac{c + d\sqrt{-1}}{a + b\sqrt{-1}} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}\sqrt{-1}.$$

If we agree that the abscissa represents the real part of the complex variable and the ordinate the imaginary part, then a line OP (Fig. 47) can be represented by $x+y\sqrt{-1}$. The length of OP, $\sqrt{x^2+y^2}$, is the modulus of the complex variable and its inclination to the axis of x is $\tan^{-1}\frac{y}{x}$. If r and θ be the polar coordinates of P, then

Argand's method of representing a complex variable.

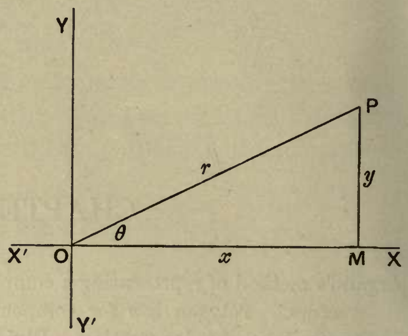

Fig. 47. OP is the graphical representation of the complex variable $x+y\sqrt{-1}$.

$$r=\sqrt{x^2+y^2},$$

and
$$\tan\theta=\frac{y}{x}.$$

Vectors.

If OP rotate about the point P with uniform angular velocity ω and if OX be the initial position of OP, then

$$\theta=\omega t,$$
$$OM=r\cos\omega t,$$
$$PM=r\sin\omega t,$$

where t is the time in seconds since OP coincided with OX. The line OP is called a vector and may be represented by

$$r\cos\omega t+\sqrt{-1}\,r\sin\omega t.$$

By trigonometry this expression may be written in the form $r\epsilon^{\omega t\sqrt{-1}}$ where ϵ is the base of Neperian logarithms.

Addition of vectors.

Two vectors OP and OQ are compounded by the parallelogram law. To prove this, suppose that OP represents $x_1+y_1\sqrt{-1}$, and that OQ represents $x_2+y_2\sqrt{-1}$.

Now, if OR (Fig. 48) represents $X+Y\sqrt{-1}$, we see at once by projections that

$$X=x_1+x_2 \text{ and } Y=y_1+y_2.$$

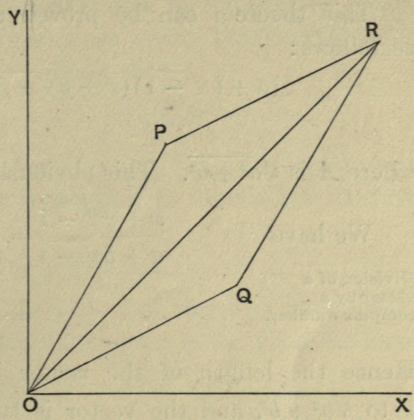

Fig. 48.

Hence OR represents

$$x_1 + x_2 + (y_1 + y_2)\sqrt{-1},$$

and may be called the resultant of the addition of the vectors represented by OP and OQ. If R be the magnitude of the resultant and θ its inclination to the axis of X, then

$$R = \{(x_1 + x_2)^2 + (y_1 + y_2)^2\}^{\frac{1}{2}},$$

and $$\tan\theta = \frac{y_1 + y_2}{x_1 + x_2}.$$

Polygon law for compounding vectors.

Proceeding in exactly the same way as in the last paragraph, we can show that the equations

$$R = \{(\Sigma x)^2 + (\Sigma y)^2\}^{\frac{1}{2}}$$

and $$\tan\theta = \frac{\Sigma y}{\Sigma x}$$

give the magnitude and the inclination of the resultant of any number of vectors.

Multiplication of a vector by a complex number.

Suppose that the vector is represented by $x + y\sqrt{-1}$ and that the number is $a + b\sqrt{-1}$. Then by ordinary multiplication we find that the result of the operation is a vector whose length is

$$\sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

and its inclination to the axis

$$\tan^{-1}\frac{bx + ay}{ax - by} = \theta + \alpha,$$

where $$\tan\theta = \frac{y}{x} \text{ and } \tan\alpha = \frac{b}{a}.$$

The result of the operation is therefore to multiply the length of the vector by $\sqrt{a^2 + b^2}$ and to advance the vector through an angle α, where $\tan\alpha = \frac{b}{a}$.

This theorem can be proved more simply trigonometrically as follows:

$$(a+b\sqrt{-1})(x+y\sqrt{-1}) = A\epsilon^{\alpha\sqrt{-1}}\, r\epsilon^{\theta\sqrt{-1}},$$
$$= Ar\epsilon^{(\alpha+\theta)\sqrt{-1}},$$

where A is $\sqrt{a^2+b^2}$. This obviously proves the theorem.

Division of a vector by a complex number.

We have
$$\frac{x+y\sqrt{-1}}{a+b\sqrt{-1}} = \frac{r\epsilon^{\theta\sqrt{-1}}}{A\epsilon^{\alpha\sqrt{-1}}}$$
$$= \frac{r}{A}\epsilon^{(\theta-\alpha)\sqrt{-1}}.$$

Hence the length of the vector is diminished in the ratio of 1 to $\sqrt{a^2+b^2}$, and the vector is turned back through an angle $\tan^{-1}\frac{b}{a}$.

Application to the theory of alternating currents.

We have seen (page 41) that the equation for the current in a simple inductive circuit is

$$e = Ri + L\frac{di}{dt}.$$

This may be written in the form

$$e = (R+LD)\,i,$$

where D is an operator. If i be an harmonic function $I\sin\omega t$, then

$$D\,.\,I\sin\omega t = + \omega I\cos\omega t,$$

and
$$D^2\,.\,I\sin\omega t = -\omega^2 I\sin\omega t.$$

Thus
$$D^2 = -\omega^2,$$

and
$$D = \omega\sqrt{-1}.$$

Similarly, if i were $I\cos\omega t$, D would have the same value. Now if we suppose that both e and i are complex quantities and can be represented by vectors, then by the preceding theorems the operations of multiplying or dividing by the complex number $R+L\omega\sqrt{-1}$ can easily be performed and the real part of the result will give the required solution. Although at first sight it may appear clumsy to introduce an imaginary term in the expression for i, seeing that we ultimately reject the coefficient of $\sqrt{-1}$ in our result, yet it will be found that this artifice introduces trigonometrical symmetry, and in some cases greatly simplifies the calculation. We will use square brackets to denote a vector quantity so that if $i = I\cos\omega t$, then

$$[i] = I\cos\omega t + \sqrt{-1}\, I\sin\omega t.$$

Denoting the operator $R + L\omega\sqrt{-1}$ by $[\rho]$, the equation in alternating current theory corresponding to Ohm's law in direct current theory will be

$$[e] = [\rho][i] \quad \text{.........................} (1).$$

To prove this it is sufficient to notice that since, by hypothesis, i and therefore also e is an harmonic function of the time, we have

$$E\cos(\omega t + \alpha) = [\rho]\, I\cos(\omega t + \beta),$$

therefore, by differentiating and multiplying by $-\dfrac{\sqrt{-1}}{\omega}$, we get

$$\sqrt{-1}\, E\sin(\omega t + \alpha) = [\rho]\sqrt{-1}\, I\sin(\omega t + \beta).$$

By adding these equations we arrive at (1). We can use this equation to find e.

We have

$$\begin{aligned}[e] &= (R + L\omega\sqrt{-1})(I\cos\omega t + \sqrt{-1}\, I\sin\omega t)\\ &= \sqrt{R^2 + L^2\omega^2}\,\epsilon^{\alpha\sqrt{-1}}\, I\epsilon^{\omega t\sqrt{-1}}\\ &= \sqrt{R^2 + L^2\omega^2}\, I\epsilon^{(\omega t + \alpha)\sqrt{-1}}\\ &= \sqrt{R^2 + L^2\omega^2}\, I\{\cos(\omega t + \alpha) + \sqrt{-1}\sin(\omega t + \alpha)\},\end{aligned}$$

therefore $e = \sqrt{R^2 + L^2\omega^2}\, I\cos(\omega t + \alpha)$.

Similarly if

$$[e] = E\cos\omega t + \sqrt{-1}\, E\sin\omega t,$$

then
$$\begin{aligned}[i] &= \frac{[e]}{[\rho]}\\ &= \frac{E}{\sqrt{R^2 + L^2\omega^2}}\,\epsilon^{(\omega t - \alpha)\sqrt{-1}}.\end{aligned}$$

Thus

$$i = \frac{E\sin(\omega t - \alpha)}{\sqrt{R^2 + L^2\omega^2}}$$

where

$$\tan\alpha = \frac{L\omega}{R}.$$

It will be noted that this method only gives us the particular integral of the equation

$$e = Ri + L\frac{di}{dt},$$

in the special case when e is a simple harmonic function. To get the complete integral we have to add on the term $B\epsilon^{-\frac{R}{L}t}$. This term however is as a rule only important for a fraction of a second after switching on (see page 42).

The currents in a divided circuit.

If the current in each branch be $\dot{i}_1$ and $\dot{i}_2$ (Fig. 49), and the resistances be r_1 and r_2, then for direct currents,

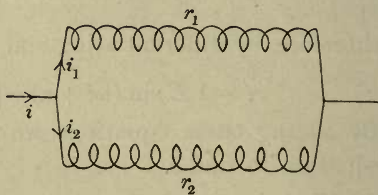

Fig. 49. Calculation of harmonic currents in a divided circuit.

$$\dot{i}_1 = \frac{r_2}{r_1 + r_2}\dot{i},$$

$$\dot{i}_2 = \frac{r_1}{r_1 + r_2}\dot{i},$$

and $$R = \frac{r_1 r_2}{r_1 + r_2},$$

where i is the current in the main and R the resistance of the two branches in parallel.

With alternating currents, if we denote the impedance of the branches by Z_1, Z_2, and if Z_1 equals $\sqrt{R_1^2 + L_1^2\omega^2}$ and $\tan\alpha_1$ equals $\frac{L_1\omega}{R_1}$, then

$$[\dot{i}_1] = \frac{[\rho_2]}{[\rho_1] + [\rho_2]}\cdot[\dot{i}],$$

$$= \frac{Z_2\epsilon^{\alpha_2\sqrt{-1}}}{Z'\epsilon^{\alpha'\sqrt{-1}}} I\epsilon^{\omega t\sqrt{-1}},$$

where $Z' = \{(R_1 + R_2)^2 + \omega^2(L_1 + L_2)^2\}^{\frac{1}{2}}$ and $\tan\alpha' = \frac{\omega(L_1 + L_2)}{R_1 + R_2}$.

Thus $$\dot{i}_1 = \frac{\{R_2^2 + \omega^2 L_2^2\}^{\frac{1}{2}} I}{\{(R_1 + R_2)^2 + \omega^2(L_1 + L_2)^2\}^{\frac{1}{2}}}\cos(\omega t + \alpha_2 - \alpha'),$$

and $$\dot{i}_2 = \frac{\{R_1^2 + \omega^2 L_1^2\}^{\frac{1}{2}} I}{\{(R_1 + R_2)^2 + \omega^2(L_1 + L_2)^2\}^{\frac{1}{2}}}\cos(\omega t + \alpha_1 - \alpha').$$

If $\frac{L_1}{R_1} = \frac{L_2}{R_2}$, then $\alpha_1 = \alpha_2 = \alpha'$, and i_1 and i_2 are in phase with i. If $\frac{L_1}{R_1}$ is greater than $\frac{L_2}{R_2}$, then α_1 is greater than α_2, and α' is greater than α_2, but less than α_1. Hence the current i_1 lags behind i, but the phase of the current i_2 is in advance of i.

Again, since $[e] = [\rho_1][i_1] = [\rho_2][i_2]$, we have

$$[\rho] = \frac{[\rho_1][\rho_2]}{[\rho_1] + [\rho_2]},$$

and
$$R + L\omega\sqrt{-1} = \frac{Z_1 Z_2}{Z'} \epsilon^{(\alpha_1 + \alpha_2 - \alpha')\sqrt{-1}}$$

$$= \frac{Z_1 Z_2}{Z'}\{\cos(\alpha_1 + \alpha_2 - \alpha') + \sqrt{-1}\sin(\alpha_1 + \alpha_2 - \alpha')\}.$$

Thus
$$R = \frac{Z_1 Z_2}{Z'}\cos(\alpha_1 + \alpha_2 - \alpha'),$$

and
$$L\omega = \frac{Z_1 Z_2}{Z'}\sin(\alpha_1 + \alpha_2 - \alpha'),$$

and
$$Z = \frac{Z_1 Z_2}{Z'},$$

where R is the equivalent resistance, L the equivalent inductance, and Z the equivalent impedance of the two branches in parallel. We see that i lags behind e by an angle $\alpha_1 + \alpha_2 - \alpha'$.

The above formulae for the effective resistance and inductance of the branched circuit may be written in the form

$$R = \frac{R_1 R_2}{R_1 + R_2} + \frac{\omega^2}{R_1 + R_2} \cdot \frac{(L_1 R_2 - L_2 R_1)^2}{Z'^2},$$

$$L = \frac{L_1 R_2^2 + L_2 R_1^2}{(R_1 + R_2)^2} - \frac{(L_1 R_2 - L_2 R_1)^2}{L_1 + L_2}\left\{\frac{1}{(R_1 + R_2)^2} - \frac{1}{Z'^2}\right\},$$

where $Z' = \{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2\}^{\frac{1}{2}}$.

Hence, as the frequency increases, the apparent resistance increases from $\frac{R_1 R_2}{R_1 + R_2}$, its minimum value, to $\frac{L_2^2 R_1 + L_1^2 R_2}{(L_1 + L_2)^2}$, its maximum value, and the apparent self inductance diminishes from $\frac{L_1 R_2^2 + L_2 R_1^2}{(R_1 + R_2)^2}$, its maximum value, to $\frac{L_1 L_2}{L_1 + L_2}$, its minimum value.

It is also worth noting that when the time constants of the circuits are equal, R and L are independent of the frequency.

Inductive coil in series with choking coil (o, L) shunted by a non-inductive resistance x.

Suppose (Fig. 50) that we maintain the effective value of the P. D. between A and B constant. Suppose also that the resistance of the inductive coil is r, and its self inductance l, then by the preceding formulae,

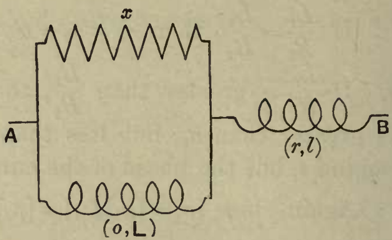

Fig. 50. Current is a maximum for a particular value of x when the effective value of the applied P. D. is constant.

$$Z^2 = \left\{\frac{x\omega^2L^2}{x^2+\omega^2L^2} + r\right\}^2 + \omega^2\left\{\frac{Lx^2}{x^2+\omega^2L^2} + l\right\}^2,$$

where Z is the impedance of the circuit AB.

Now by the differential calculus this is a maximum or a minimum when

$$x^2 - \frac{\omega^2 L(L+2l)}{r}x - \omega^2L^2 = 0,$$

i.e. when $$x = \frac{\omega^2L(L+2l)}{2r} + \frac{\omega L}{2r}\{\omega^2(L+2l)^2 + 4r^2\}^{\frac{1}{2}}.$$

Since x must be positive, we have prefixed the positive sign to the radical. It is easy to see that this value of x makes Z a maximum, and therefore the current through the inductive coil a minimum. Hence shunting a choking coil with a non-inductive resistance sometimes increases the apparent resistance of a circuit, a result which has been noticed in practical work.

The apparent resistance and inductance of branched circuits.

Let (R_1, L_1), (R_2, L_2)... be the resistances and self inductances of the n branches, then, if we neglect mutual inductance,

$$[i_1] = \frac{1}{[\rho_1]}[e],$$

$$[i_2] = \frac{1}{[\rho_2]}[e],$$

....

$$[i_n] = \frac{1}{[\rho_n]}[e].$$

If i be the current in the main, then

$$[i] = [i_1] + [i_2] + \ldots\ldots + [i_n]$$

$$= \left\{\frac{1}{[\rho_1]} + \frac{1}{[\rho_2]} + \ldots\ldots\right\}[e]$$

$$= \left\{\frac{1}{R_1 + L_1\omega\sqrt{-1}} + \frac{1}{R_2 + L_2\omega\sqrt{-1}} + \ldots\ldots\right\}[e]$$

$$= \left\{\Sigma\frac{R_1}{R_1^2 + L_1^2\omega^2} - \Sigma\frac{L_1\omega}{R_1^2 + L_1^2\omega^2}\sqrt{-1}\right\}[e]$$

$$= \frac{[e]}{R + L\omega\sqrt{-1}},$$

where R is the equivalent resistance and L the equivalent inductance of the n branches in parallel.

Therefore

$$\frac{R}{R^2 + L^2\omega^2} = \Sigma\frac{R_1}{R_1^2 + L_1^2\omega^2} = \frac{1}{a}.$$

And

$$\frac{L\omega}{R^2 + L^2\omega^2} = \Sigma\frac{L_1\omega}{R_1^2 + L_1^2\omega^2} = \frac{1}{b}.$$

Thus

$$\frac{1}{R^2 + L^2\omega^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Hence

$$R = \frac{1}{a}\Big/\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

and

$$L\omega = \frac{1}{b}\Big/\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

We also see that the current in the main lags behind the applied P.D. by an angle θ where

$$\tan\theta = \frac{a}{b}.$$

The currents in a branched circuit when mutual inductance is taken into account.

Let (R_1, L_1), (R_2, L_2) be the resistances and inductances of the two branches, and let M be their mutual inductance, then with the usual notation

$$\left.\begin{aligned} e &= (R_1 + L_1D)i_1 + MDi_2, \\ e &= (R_2 + L_2D)i_2 + MDi_1. \end{aligned}\right\} \quad \ldots\ldots\ldots (a).$$

Assuming that the functions are simple harmonic, we may write $\omega\sqrt{-1}$ for D, and solving the equations we find

$$i_1\{R_1R_2 + (M^2 - L_1L_2)\omega^2 + (L_2R_1 + L_1R_2)\omega\sqrt{-1}\} = \{R_2 + (L_2 - M)\omega\sqrt{-1}\}e,$$

$$i_2\{R_1R_2 + (M^2 - L_1L_2)\omega^2 + (L_2R_1 + L_1R_2)\omega\sqrt{-1}\} = \{R_1 + (L_1 - M)\omega\sqrt{-1}\}e.$$

Hence since i the current in the main equals $i_1 + i_2$, we get

$$e = \frac{c + d\sqrt{-1}}{a + b\sqrt{-1}}\, i$$

$$= \left\{\frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}\sqrt{-1}\right\} i$$

$$= \{R + L\omega\sqrt{-1}\}\, i,$$

where $a = R_1 + R_2$, $c = R_1R_2 + (M^2 - L_1L_2)\omega^2$,

$b = (L_1 + L_2 - 2M)\,\omega$, $d = (L_1R_2 + L_2R_1)\,\omega$.

Hence

$$R = \frac{ac + bd}{a^2 + b^2}$$

$$= \frac{R_1R_2(R_1 + R_2) + \omega^2\{(L_2 - M)^2R_1 + (L_1 - M)^2R_2\}}{(R_1 + R_2)^2 + \omega^2(L_1 + L_2 - 2M)^2},$$

$$L = \frac{L_1R_2^2 + L_2R_1^2 + 2MR_1R_2 + (L_1L_2 - M^2)(L_1 + L_2 - 2M)\,\omega^2}{(R_1 + R_2)^2 + \omega^2(L_1 + L_2 - 2M)^2}.$$

Hence, as the frequency increases, the effective resistance increases from

$$\frac{R_1R_2}{R_1 + R_2} \text{ to } \frac{(L_2 - M)^2R_1 + (L_1 - M)^2R_2}{(L_1 + L_2 - 2M)^2}.$$

It is to be noted that in the above equations M may be either positive or negative, depending on how the coils are connected with the mains. When the frequency is zero

$$L = L_1\left(\frac{R_2}{R_1 + R_2}\right)^2 + L_2\left(\frac{R_1}{R_1 + R_2}\right)^2 + 2M\frac{R_1R_2}{(R_1 + R_2)^2}.$$

If, for example, we measured the self inductance of the two coils in parallel by Maxwell's method, the self inductance L found would be the L given by this formula. A strict proof of this particular formula can be given by other methods. As the frequency increases the effective inductance L diminishes, and it has its least possible value

$$\frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

when the frequency is infinite. We have shown on page 24 that $L_1L_2 - M^2$ is zero or positive. The case when it is zero need not be considered, as it is impossible in practice to arrange two circuits so as to satisfy this condition. In practice, $L_1L_2 - M^2$ is positive, and therefore, since M is less than $\sqrt{L_1L_2}$, $L_1 + L_2 - 2M$ is positive.

The equations (a) may be written in the form

$$e = (R_1 + L_1D)i_1 + MDi_2 = (R_2 + L_2D)i_2 + MDi_1.$$

If i denote the current in the main, we must have, at every instant, i equal to the sum of i_1 and i_2. Substituting $i - i_1$ for i_2 in the above equation we get

$$(R_1 + L_1D)\,i_1 + MD\,(i - i_1) = (R_2 + L_2D)\,(i - i_1) + MDi_1$$

and thus

$$(R_1 + R_2)\,i_1 + (L_1 + L_2 - 2M)\,Di_1 = R_2i + (L_2 - M)\,Di \qquad \ldots\ldots\ldots\ldots(b).$$

If i_1 and i_2 are harmonic functions we get

$$A_1^2 = \frac{R_2^2 + \omega^2(L_2 - M)^2}{(R_1 + R_2)^2 + \omega^2(L_1 + L_2 - 2M)^2} \cdot A^2.$$

When the frequency is very high

$$\pm A_1 = \frac{L_2 - M}{L_1 + L_2 - 2M} A,$$

$$\pm A_2 = \frac{L_1 - M}{L_1 + L_2 - 2M} A.$$

We choose the signs in these formulae so that A_1 and A_2 are positive. Suppose that M is less than L_1 and L_2, then the positive signs must be chosen and we have

$$A_1 + A_2 = A.$$

Thus the phase difference ϕ between the vectors representing A_1 and A_2 is zero.

Now the average value of the electromagnetic energy stored in the field is

$$\tfrac{1}{2}L_1A_1^2+\tfrac{1}{2}L_2A_2^2+MA_1A_2\cos\phi.$$

Since $\cos\phi$ is unity and A_2 equals $A-A_1$, this becomes

$$\tfrac{1}{2}L_1A_1^2+\tfrac{1}{2}L_2(A-A_1)^2+MA_1(A-A_1).$$

If we suppose that A is a constant, then this equation is a function of A_1 and attains its minimum value when

$$A_1=\frac{L_2-M}{L_1+L_2-2M}A.$$

Hence, with very high frequencies, we see that in this case the values of the currents are such that the average value of the energy stored in the field is a minimum.

When M is greater than L_2 but less than L_1, then

$$-A_1+A_2=A.$$

Thus the phase difference ϕ between the vectors of A_1 and A_2 is 180°. Proceeding as before we find that the average value of the energy stored in the field is again a minimum. Therefore since M cannot be greater than L_1 and L_2 we see that this theorem is true in all cases.

The above theorems may also be deduced from the following more general theorem. In the particular case of high frequency currents the terms $(R_1+R_2)\,i_1$ and $R_2 i$ in equation (b) can be neglected in comparison with the other terms, we thus get

$$(L_1+L_2-2M)\,Di_1=(L_2-M)\,Di$$

and integrating

$$(L_1+L_2-2M)\,i_1=(L_2-M)\,i+\text{constant}.$$

The constant must vanish, for i_1 and i are periodic functions of the time the mean values of which are zero, and thus

$$(L_1+L_2-2M)\,i_1=(L_2-M)\,i$$

and similarly $$(L_1+L_2-2M)\,i_2=(L_2-M)\,i.$$

It is to be noted that in proving these equations we have made no assumption as to the shape of the applied wave of electromotive force.

Now the value of the energy stored in the field, at the instant when the currents are i_1 and i_2 respectively, is

$$\tfrac{1}{2} L_1 i_1^2 + \tfrac{1}{2} L_2 i_2^2 + M i_1 i_2$$

or

$$\tfrac{1}{2} L_1 i_1^2 + \tfrac{1}{2} L_2 (i - i_1)^2 + M i_1 (i - i_1),$$

where i is the current in the main. For a given value of i this has its minimum value when

$$(L_1 + L_2 - 2M)\, i_1 = (L_2 - M)\, i.$$

Since this is the actual value of i_1 we see that with high frequency currents, the main current splits up in such a manner that the energy stored in the field is a minimum.

Condition that the energy expended have a stationary value.

Let U denote the energy expended in heating the branches, from the moment of closing the switch to the time t, together with the energy stored in the field at this instant. Then we have

$$U = \tfrac{1}{2} L_1 i_1^2 + M i_1 (i - i_1) + \tfrac{1}{2} L_2 (i - i_1)^2 + \int_0^t \{R_1 i_1^2 + R_2 (i - i_1)^2\}\, dt.$$

Now if δU be the increment of U due to an alteration in the distribution of the current in the two branches, we have

$$\delta U = L_1 i_1 \delta i_1 + M i_2 \delta i_1 - M i_1 \delta i_1 - L_2 i_2 \delta i_1 + \int_0^t \{2R_1 i_1 \delta i_1 - 2R_2 i_2 \delta i_1\}\, dt.$$

By the calculus of variations the condition for a maximum or a minimum value of U is that the coefficients of δi_1 inside and outside of the integral sign must vanish simultaneously, and hence we get

$$(L_1 - M)\, i_1 = (L_2 - M)\, i_2 \quad \text{and} \quad R_1 i_1 = R_2 i_2$$

as the required conditions. We see therefore that when

$$\frac{L_1 - M}{R_1} = \frac{L_2 - M}{R_2}$$

U has a maximum or a minimum value.

It is interesting to note that

$$R_1 i_1 = R_2 i_2$$

is the condition that the heating effect at a given instant on the branch conductors is a minimum, and that

$$(L_1 - M)\, i_1 = (L_2 - M)\, i_2$$

is the condition that the energy stored in the field is a minimum.

Important consequences in this case.

In practice, when M is less than L_1 and L_2, it is easy to adjust the ratio of the resistances of the arms so that

$$\frac{R_1}{R_2} = \frac{L_1 - M}{L_2 - M}.$$

We shall now prove that in this case, from the moment of closing the switch, the current waves in the branches are similar to one another and are therefore also similar to the current in the main. From the equations (a) we see that

$$R_1 i_1 + (L_1 - M)\, Di_1 = R_2 i_2 + (L_2 - M)\, Di_2,$$

and therefore $$R_1 i_1 - R_2 i_2 = -\frac{L_1 - M}{R_1}\frac{d}{dt}(R_1 i_1 - R_2 i_1).$$

Solving this equation we get

$$R_1 i_1 - R_2 i_2 = A\epsilon^{-\frac{R_1}{L_1 - M}t},$$

where A is a constant. Now at the instant when t is zero both i_1 and i_2 must be zero, otherwise we should have a finite amount of energy stored in the field in an infinitely short time. Thus A is zero and so

$$R_1 i_1 - R_2 i_2 = 0.$$

We see therefore that i_1 and i_2 are similar waves whatever may be the shape of the applied potential difference wave. We see also that

$$i_1 = \frac{R_2}{R_1 + R_2}\, i \quad \text{and} \quad i_2 = \frac{R_1}{R_1 + R_2}\, i.$$

It is evident that i_1, i_2 and i vanish simultaneously, but the applied potential difference e does not vanish at this instant. It is to be noted that for a given effective value of the applied potential difference, the values of A_1, A_2 and A depend on the shape of the wave and the frequency, but their ratios are always constant. This theorem may be usefully applied in the design of alternating current ammeters.

Graphical solution. The formulae given above for the effective values of R, L, A_1 and A_2 may also be derived from the following graphical constructions.

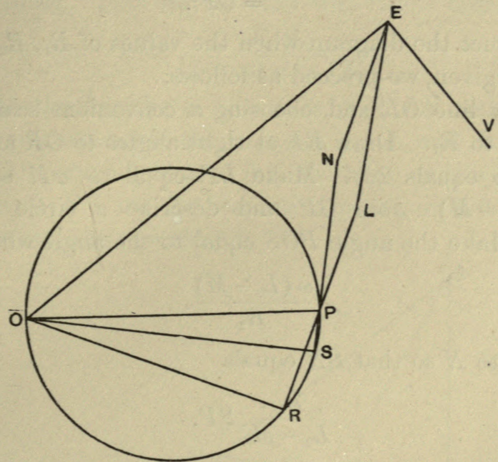

Fig. 51. Currents in a divided circuit.

$OE = V$; $OS = R_2 A_2$; $OR = R_1 A_1$.

M is less than L_1 or L_2.

The equations (a) given above show that the vector of the effective value of the applied potential difference V is the resultant of the component vectors $R_1 A_1$, $\omega L_1 A_1$ and $\omega M A_2$ and also, that it is the resultant of $R_2 A_2$, $\omega L_2 A_2$ and $\omega M A_1$. The vectors $\omega L_1 A_1$ and $\omega M A_2$ are each perpendicular to $R_1 A_1$ and the vectors $\omega L_2 A_2$ and $\omega M A_1$ are each perpendicular to $R_2 A_2$.

In Fig. 51 M is less than either L_1 or L_2.

$$OE = \text{the applied P.D.} = V,$$

$$OR = R_1 A_1 \qquad OS = R_2 A_2,$$

$$RL = \omega L_1 A_1 \qquad SN = \omega L_2 A_2,$$

$$LE = \omega M A_2 \qquad NE = \omega M A_1.$$

Now PN and LE are parallel, and so also are NE and PL. Therefore $NPLE$ is a parallelogram and thus PL equals $\omega M A_1$ and PN equals $\omega M A_2$. Hence PR is $\omega (L_1 - M) A_1$ and PS is $\omega (L_2 - M) A_2$. Since the angles at R and S are right angles, the circle described on OP as diameter passes through R and S. Hence

$$(R_1A_1)^2+\{\omega(L_1-M)A_1\}^2=(R_2A_2)^2+\{\omega(L_2-M)A_2\}^2$$
$$=OP^2.$$

To construct the diagram when the values of R_1, R_2, L_1, L_2, M, f, and V are given, we proceed as follows.

Draw any line OR and, choosing a convenient scale, make its length equal to R_1. Draw RL at right angles to OR and equal to ωL_1 where ω equals $2\pi f$. Make LP equal to ωM so that RP equals $\omega(L_1-M)$. Join OP and describe a circle on OP as diameter. Make the angle POS equal to the angle whose tangent is

$$\frac{\omega(L_2-M)}{R_2}.$$

Produce SP to N so that SN equals

$$\frac{L_2}{L_2-M}\cdot SP.$$

Through N and L draw lines NE and LE parallel to RL and SN respectively. Then OE will be the effective value of the applied potential difference which produces unit current in the branch (R_1, L_1). Now the phase differences and the relative magnitudes of the various vectors are independent of the absolute value of the applied potential difference provided that the frequency does not alter. Hence if we choose the scale of the diagram so that OE represents V, then, on this scale, OR will represent R_1A_1 and OS will represent R_2A_2 in magnitude and phase, and thus, by dividing these lengths by R_1 and R_2 respectively, we get A_1 and A_2.

If ϕ be the phase difference between A_1 and A_2 and $\tan\alpha$ and $\tan\beta$ be equal to

$$\omega\frac{L_1-M}{R_1} \text{ and } \omega\frac{L_2-M}{R_2}$$

respectively, we have

$$\tan\phi=\tan(\alpha-\beta)$$
$$=\frac{\omega\dfrac{L_1-M}{R_1}-\omega\dfrac{L_2-M}{R_2}}{1+\omega^2\dfrac{L_1-M}{R_1}\cdot\dfrac{L_2-M}{R_2}} \quad \text{..................(c).}$$

We can see at once from equation (c) that when the frequency is very low the angle of phase difference between A_1 and A_2 is very small. When the frequency is very high, the phase difference is again very small. It is easy to show that it is a maximum when

$$\omega^2 = \frac{R_1 R_2}{(L_1 - M)(L_2 - M)} \quad \text{.....................} (d).$$

If we divide the lengths of the lines NE, LE and PE by ωM we get the magnitudes of the currents A_1, A_2 and A. If we draw EV (Fig. 51) perpendicular to OE, then the angles NE, LE and PE make with EV are the respective phase differences between the currents and the applied P.D.

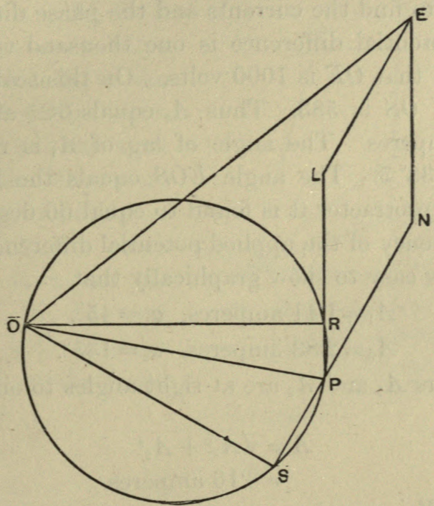

Fig. 52. Currents in a divided circuit.
OE = applied P.D.; $OR = R_1 A_1$; $OS = R_2 A_2$.
M is greater than L_1 and less than L_2.

Fig. 52 gives the graphical construction when M is greater than L_1 and less than L_2. In this case A_1 and A_2 are in quadrature when

$$\omega^2 = \frac{-R_1 R_2}{(L_1 - M)(L_2 - M)}$$

and the phase difference increases as the frequency increases. When the frequency is high, the currents are nearly in opposition in phase, and hence the current in the main is nearly equal to their difference. If L_1 be approximately equal to L_2 then the current in the main may be very much smaller than the branch currents. The inductive effects produce as it were a whirlpool of current in the two branches. Large circulating currents produced in this way are often useful in alternating current work.

The diagram in Fig. 52 is drawn to scale for the case of two coils (100, 0·5), (100, 1) the mutual inductance being 0·6 henry and the frequency being 16, so that ω is nearly 100. On the given scale, OR is equal to 100, RL equals 50 and RP equals 10. A circle is described on OP as diameter. The angle POS equals the angle whose tangent is 0·4 and SN equals $\frac{5}{2}SP$. The parallelogram $PNEL$ is then completed and OE is joined.

If we wish to find the currents and the phase differences when the applied potential difference is one thousand volts, we choose a new scale so that OE is 1000 volts. On this scale OR is found to be 625 and OS is 583. Thus A_1 equals 6·25 amperes and A_2 equals 5·83 amperes. The angle of lag of A_1 is the angle EOR and it equals 38°·5. The angle EOS equals the lag of A_2 and measured by a protractor it is found to equal 66 degrees.

If the frequency of the applied potential difference be increased to 80, then it is easy to show graphically that

$$A_1 = 1·41 \text{ amperes}, \quad \alpha_1 = 45°,$$
$$A_2 = 2·83 \text{ amperes}, \quad \alpha_2 = 135°.$$

Thus the vectors A_1 and A_2 are at right angles to one another and so

$$A = \sqrt{A_1^2 + A_2^2}$$
$$= 3·16 \text{ amperes.}$$

In Fig. 53 M is negative.

It follows from (c) that ϕ is zero when ω is zero and again when ω is infinite.

The angle ϕ has its maximum value when the frequency is determined by (d). The diagram in Fig. 53 is worked out for the case of two coils (53, 0·29), (117, 0·18) and M equal to $-0·14$. The frequency has been taken equal to 80 so that ω is nearly equal to 500. We find by measuring the lines and angles that

$$A_1 = 9·3 \text{ amperes}, \quad \alpha_1 = 40°,$$
$$A_2 = 10·2 \text{ amperes}, \quad \alpha_2 = 18°,$$
$$A = 19·2 \text{ amperes}, \quad \text{Impedance} = 52 \text{ ohms}.$$

It is to be noted that all the theorems and formulae in this Chapter, deduced by the method of the complex variable, are proved on the assumption of a sine wave of E.M.F. Moreover they only give the particular integrals of the differential equations involved, and hence it is best to regard them merely as giving a first approximate solution of the general problem. As a rule, it is better for engineers to solve problems graphically, as the graphical method is quite accurate enough and far more instructive than the mere algebraical manipulation of complex quantities in which the physical laws are not so apparent. In the next Chapter we will consider the graphical method and its limitations.

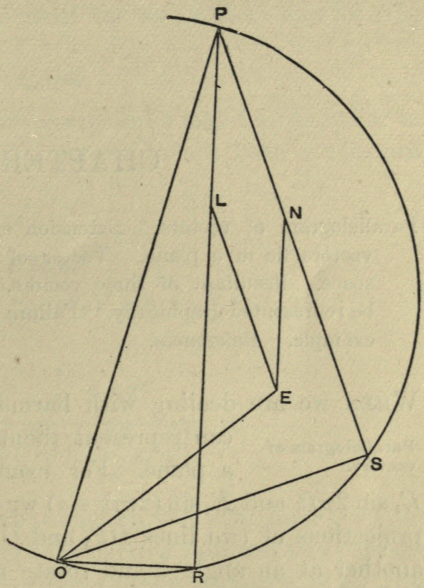

Fig. 53. Currents in a divided circuit. OE = applied P.D.; $OR = R_1 A_1$; $OS = R_2 A_2$. M is negative.

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CHAPTER VIII.

Parallelogram of vectors. Extension of theorem. Condition that three vectors lie in a plane. Vector of a constant quantity. Vectors in space. Resultant of three vectors. Condition that four vectors can be represented graphically. Failure of graphical methods. Numerical example. References.

Parallelogram of vectors.

WHEN we are dealing with harmonically varying quantities, we can represent them graphically by lines drawn in a plane. For example, if we have two E.M.F.'s $E_1 \sin 2\pi ft$ and $E_2 \sin(2\pi ft + \alpha)$ we see that their values are the projections of two lines OP_1 and OP_2, which are inclined to one another at an angle α and rotate round O 'f' times per second, upon a fixed line OA drawn at right angles to the initial position of OP_1. The resultant got by adding the two E.M.F.'s together, $E_1 \sin 2\pi ft + E_2 \sin(2\pi ft + \alpha)$, is easily shown to be the projection upon OA of the diagonal OR of the parallelogram constructed with OP_1 and OP_2 for adjacent sides. Since in practice we are generally only concerned with the R.M.S. values of the E.M.F.'s and their phase differences, and since the R.M.S. value of a simple harmonic quantity bears a constant ratio to its maximum value, it is usual to represent the E.M.F.'s by lines OP_1, OP_2 and OR equal to their R.M.S. values and inclined to one another at angles equal to their phase differences. These lines are called vectors and the above theorem is the parallelogram of vectors. It can obviously be generalised into the polygon of vectors.

Extension of theorem.

In alternating current work we seldom have to do with harmonic waves of P.D. and current, but still graphical methods are found to be convenient in many cases. We have therefore to investigate the limits of accuracy of these methods. Let us first consider the case of two periodic quantities, for example alternating electromotive forces. Let e_1 and e_2 be the instantaneous values of the two E.M.F.'s and let e be their resultant, then

$$e = e_1 + e_2.$$

Now e_1 and e_2 may be dissimilar curves and e may be dissimilar to both of them, hence their R.M.S. values do not bear a constant ratio to their maximum values.

In all cases however we have

$$e^2 = e_1^2 + e_2^2 + 2e_1e_2,$$

and thus

$$V^2 = V_1^2 + V_2^2 + 2V_1V_2\cos\alpha_{1.2},$$

where

$$\cos\alpha_{1.2} = \frac{\int_0^T e_1e_2dt}{\left\{\int_0^T e_1^2dt \cdot \int_0^T e_2^2dt\right\}^{\frac{1}{2}}} \quad \text{...............(1).}$$

Hence the parallelogram construction will still hold good for compounding lines representing the effective values of two periodic quantities if we define phase difference by equation (1). This definition we have already given in Chapter VI. It is customary to call the lines representing V_1, V_2 and V vectors. The two vectors and their resultant are lines in one plane and the angles between the various vectors will give their phase differences in accordance with definition (1).

Condition that three vectors lie in a plane.

In general if a linear relation hold between the instantaneous values of three periodic alternating functions e_1, e_2 and e_3, each of which has the same period, then their R.M.S. values can be represented by lines drawn in one plane, the angles between the lines being the phase differences. For example, suppose that at every instant

$$le_1 + me_2 + ne_3 = 0,$$

where l, m and n are constants and e_1, e_2 and e_3 are periodic functions of the time, then

$$l^2e_1^2 = m^2e_2^2 + n^2e_3^2 + 2mne_2e_3,$$

thus

$$l^2V_1^2 = m^2V_2^2 + n^2V_3^2 + 2mnV_2V_3\cos\alpha_{1.3}.$$

Hence lV_1 is equal in magnitude to the resultant got by compounding two lines inclined to one another at an angle $\alpha_{2.3}$ and the lengths of which are mV_2 and nV_3 respectively, by the parallelogram construction. The line representing lV_1 must however be drawn in the opposite direction to their resultant, since

$$le_1 = -(me_2 + ne_3).$$

We also have $$\alpha_{1.2} + \alpha_{2.3} + \alpha_{3.1} = 2\pi$$
and, just as in statics,

$$\frac{lV_1}{\sin \alpha_{1.2}} = \frac{mV_2}{\sin \alpha_{2.3}} = \frac{nV_3}{\sin \alpha_{3.1}}.$$

These lines lV_1, mV_2 and nV_3 are called vectors in engineering practice, but the instantaneous values of e_1, e_2 and e_3 can no longer be represented by the projections of lines rotating with constant angular velocity.

Vector of a constant quantity. It follows from the definition of phase difference given in (1) that the cosine of the phase difference between a constant and a periodic quantity is always zero, and hence the phase difference is ninety degrees. Hence the R.M.S. value of the sum of an alternating and direct current, for example, is represented in magnitude and phase by the diagonal of the rectangle constructed with the R.M.S. values of the direct and alternating components as adjacent sides. If however we have two alternating current components which are not in the same phase, then the line representing the direct current component must be drawn at right angles to the plane containing the two lines representing the alternating current components so as to be at right angles to the three lines representing the two alternating components and their resultant. We must therefore, if we are going to use graphical methods in this case, have recourse to solid geometry.

Vectors in space. In general, when we have three periodic functions and there is no linear relation connecting them, we can represent their R.M.S. values graphically by three lines drawn in space, the angles between the lines being the phase differences as determined by equation (1). In order to prove this we have to show that α, β and γ, the angles of phase difference, can always form a solid angle. We have to prove therefore that $\alpha+\beta+\gamma$ can never be greater than 4π, and also that no two of the angles can be greater than the third.

Let x, y and z be the three periodic functions and let α be the phase difference between y and z, β between z and x and γ between x and y respectively.

From the definition of phase difference we have

$$\cos\alpha = \frac{\int_0^T yz\,dt}{\left\{\int_0^T y^2 dt \,.\int_0^T z^2 dt\right\}^{\frac{1}{2}}}.$$

Divide the period T into a large number (n) of equal intervals, and let $x_1, x_2, \ldots x_n$; $y_1, y_2, \ldots y_n$; $z_1, z_2, \ldots z_n$ be the values of the functions at the end of successive intervals.

Then from the meaning of integration we have, when n is infinitely large,

$$\cos^2\alpha = \frac{(y_1z_1+y_2z_2+\ldots+y_nz_n)^2}{(y_1^2+y_2^2+\ldots+y_n^2)(z_1^2+z_2^2+\ldots+z_n^2)},$$

with corresponding values for $\cos^2\beta$ and $\cos^2\gamma$.

Let $X = x_1^2+x_2^2+\ldots+x_n^2,\quad A = y_1z_1+y_2z_2+\ldots+y_nz_n,$

$Y = y_1^2+y_2^2+\ldots+y_n^2,\quad B = z_1x_1+z_2x_2+\ldots+z_nx_n,$

$Z = z_1^2+z_2^2+\ldots+z_n^2,\quad C = x_1y_1+x_2y_2+\ldots+x_ny_n.$

Then $\quad 1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$

$$=\frac{XYZ-XA^2-YB^2-ZC^2+2ABC}{XYZ}\ldots\ldots\ldots\ldots\ldots(a).$$

Now $\quad X(XYZ-XA^2-YB^2-ZC^2+2ABC)$

$$=(XY-C^2)(XZ-B^2)-(XA-CB)^2.$$

Also $\quad XY-C^2=(x_1y_2-x_2y_1)^2+(x_1y_3-x_3y_1)^2+\ldots\ldots\ldots\ldots(b),$

$XZ-B^2=(x_1z_2-x_2z_1)^2+(x_1z_3-x_3z_1)^2+\ldots\ldots\ldots\ldots(c),$

$XA-CB=(x_1y_2-x_2y_1)(x_1z_2-x_2z_1)$

$\qquad+(x_1y_3-x_3y_1)(x_1z_3-x_3z_1)+\ldots(d).$

Now it is easy to show that

$$(P^2+Q^2+\ldots)(p^2+q^2+\ldots)$$

is not less than $(Pp + Qq + \dots)^2$. It therefore follows from (*b*), (*c*) and (*d*) that

$$(XY - C^2)(XZ - B^2) - (XA - CB)^2$$

is never negative. Hence from (*a*)

$$1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma$$

is never negative. Now the trigonometrical expression can be written in the form

$$\{\cos\gamma - \cos(\alpha+\beta)\}\{\cos(\alpha-\beta) - \cos\gamma\}.$$

Hence we see that if it vanishes, the sum of two of the angles equals the third or the sum of the three equals four right angles. The three vectors in this case are therefore in one plane.

It may also be written in the form

$$4\sin\frac{\alpha+\beta+\gamma}{2}\cdot\sin\frac{\beta+\gamma-\alpha}{2}\cdot\sin\frac{\gamma+\alpha-\beta}{2}\cdot\sin\frac{\alpha+\beta-\gamma}{2}\quad\dots(e).$$

By definition, α, β and γ lie between 0 deg. and 180 deg. As the expression (*e*) is positive, all the terms may be positive, or two of them may be negative, or all four of them may be negative.

If they are all positive, then $\alpha+\beta+\gamma$ is less than four right angles, and also any two of the angles are together greater than the third. Suppose, now, that the first two terms of (*e*) are negative. Then $\alpha+\beta+\gamma$ is greater than 2π, and $\alpha-\beta-\gamma$ is greater than zero. Therefore α is greater than π, which is contrary to the definition. Similarly, no other two terms can be negative and *à fortiori* the whole four cannot be negative. In all cases, therefore, we see that the sum of the three angles is not greater than four right angles, and that the sum of two of them is never greater than the third. Hence the three angles can always form a solid angle.

Resultant of three vectors.

We shall apply the above theorem to find graphically the effective value of the sum of three alternating periodic functions. Let e_2, e_3 and e_4 be the three functions, and let e_1 be their resultant, then

$$e_1 = e_2 + e_3 + e_4,$$

and $$e_1^2 = e_1e_2 + e_1e_3 + e_1e_4,$$

thus $$V_1^2 = V_1V_2\cos\alpha_{1.2} + V_1V_3\cos\alpha_{1.3} + V_1V_4\cos\alpha_{1.4}.$$

$$\left.\begin{aligned} \text{Hence}\quad V_1 &= V_2\cos\alpha_{1.2} + V_3\cos\alpha_{1.3} + V_4\cos\alpha_{1.4} \\ \text{similarly}\quad V_1\cos\alpha_{1.2} &= V_2 + V_3\cos\alpha_{2.3} + V_4\cos\alpha_{2.4} \\ V_1\cos\alpha_{1.3} &= V_2\cos\alpha_{2.3} + V_3 + V_4\cos\alpha_{3.4} \\ V_1\cos\alpha_{1.4} &= V_2\cos\alpha_{2.4} + V_3\cos\alpha_{3.4} + V_4 \end{aligned}\right\}\ldots\ldots(a).$$

And
$$V_1^2 = V_2^2 + V_3^2 + V_4^2 + 2V_3V_4\cos\alpha_{3.4} + 2V_4V_2\cos\alpha_{2.4} + 2V_2V_3\cos\alpha_{2.3} \quad\ldots\ldots\ldots\ldots(b).$$

In these equations V_n is the effective value of e_n and $\alpha_{n.m}$ is the phase difference between e_n and e_m.

Construct the solid angle at O (see Fig. 54), so that the angles POQ, QOR and ROP equal $\alpha_{2.3}$, $\alpha_{3.4}$ and $\alpha_{4.2}$ respectively. Complete the parallelopiped $OPQRS$ and let OS be the diagonal. Then from (*b*) we see that OS equals V_1. Hence from the equations (*a*) we see that the angles POS, QOS and ROS equal $\alpha_{1.2}$, $\alpha_{1.3}$ and $\alpha_{1.4}$ respectively. We can thus extend graphical methods to solid geometry for the case of three periodic quantities.

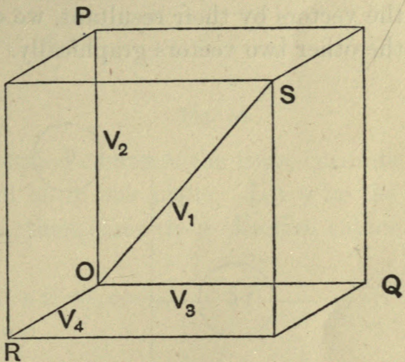

Fig. 54. *OS* is the resultant of *OR*, *OP* and *OQ*.

In this case we can show from equations (*a*) that

$$\frac{V_1}{\sin(\alpha_{2.3}, \alpha_{2.4}, \alpha_{3.4})} = \frac{V_2}{\sin(\alpha_{1.3}, \alpha_{1.4}, \alpha_{3.4})} = \frac{V_3}{\sin(\alpha_{1.2}, \alpha_{1.4}, \alpha_{2.4})} = \frac{V_4}{\sin(\alpha_{1.2}, \alpha_{1.3}, \alpha_{2.3})}$$

where $\sin(\alpha, \beta, \gamma) = \{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma\}^{\frac{1}{2}}$.

Condition that four vectors can be represented graphically.

In general when four periodic quantities of the same frequency are connected by a linear relation,

$$le_1 + me_2 + ne_3 + pe_4 = 0;$$

then their R.M.S. values can be represented in magnitude and phase by lines drawn from a point in space.

In statics these lines would represent a system of four forces in equilibrium, and we shall see later on that many statical theorems have their counterpart in electrical theory.

Failure of graphical methods.

If no linear relation connects the four periodic quantities then, as a rule, we cannot represent them graphically by lines drawn in space. For example, suppose that one of them is constant, then it would have to be represented by a line drawn at right angles to the other three, which is impossible when they form a solid angle. If however we replace two of the vectors by their resultant, we can represent this resultant and the other two vectors graphically.

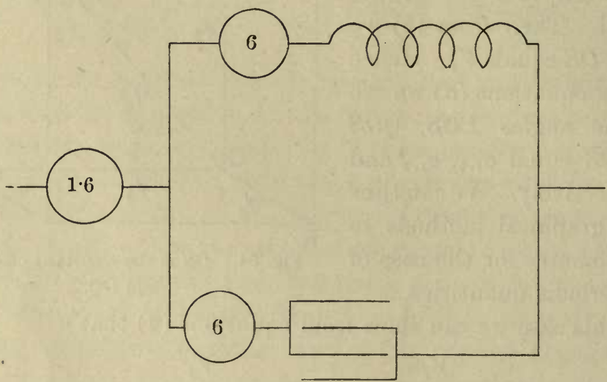

Fig. 55. Resonance of currents.

Numerical example.

A variable condenser (Fig. 55) was used to shunt a choking coil on a 2000 volt circuit; the power factor of the choking coil was 0·041 and of the condenser 0·124. The current in the choking coil was 6 amperes, and the current in the main was 1·6 amperes when the condenser was adjusted so that the current in its circuit was 6 amperes. From these data let us find the minimum possible power factor of the shunted choking coil.

Let OV (Fig. 56) represent the P.D. across the terminals of the choking coil and the condenser. Now if α and β be the phase differences between the choking coil current and the P.D. and between the condenser current and the P.D., then (Chapter VI)

$$\cos\alpha = 0{\cdot}041, \quad \cos\beta = 0{\cdot}124.$$

Thus α is 87° 39′ and β is 82° 53′.

In Fig. 56 OV represents the applied voltage, OC the choking coil current and OK the condenser current. The angle VOC is α and the angle VOK is β. Now at every instant

$$i = i_1 + i_2$$

where i is the current in the main and i_1, i_2 are the currents in the branches. Hence a linear relation connects the three currents and the three current vectors will be in one plane. Let γ be the phase difference between i_1 and i_2, then, since their effective values are each equal to 6,

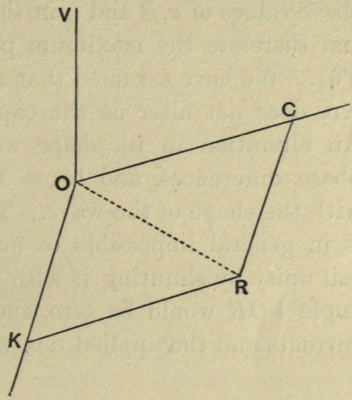

Fig. 56.

$$(1{\cdot}6)^2 = 6^2 + 6^2 + 2 \,.\, 6^2 \,.\, \cos\gamma,$$

therefore $\gamma = 164° \, 26'.$

Since $\alpha + \beta$ equals 170° 32′ it follows that OV, OC and OK are not in one plane. But OR, which represents the current in the main, is always in the plane COK.

The diagram (Fig. 56) shows us exactly what happens when we increase or diminish OK, since R always lies on CR which is a line drawn parallel to OK. We suppose that the phase differences between the applied potential difference, the condenser current and the choking coil current remain constant for all values of the condenser current. Hence the angles forming the solid angle at O remain constant. The power factor of the combined circuit is $\cos\phi$ where ϕ is the angle VOR. Now the minimum value of ϕ is got when the planes VOR and COK are at right angles to one another. In this case, by the formulae of spherical trigonometry, we get

$$\sin\phi = \frac{\sin\omega}{\sin\gamma}$$

where $\sin\omega = 2\sqrt{\sin\sigma\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}$

and σ is half the sum of the angles α, β and γ. Substituting the above values of α, β and γ in this equation, we find that ϕ is $52° 24'$ and therefore the maximum possible value of the power factor is 0·61. We have assumed that the shape of the wave of the applied P.D. does not alter as the capacity of the condenser is altered. An alteration in its shape would alter the power factors and phase differences, and hence the angles α, β and γ would vary with the shape of the wave. The diagram (Fig. 56) shows why it is in general impossible to make the power factor of a choking coil unity by shunting it with a condenser. If it were unity the angle VOR would be zero, and the vectors representing the two currents and the applied P.D. would lie in one plane.

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CHAPTER IX.

The measurement of power. The quadrant electrometer. The electrostatic wattmeter. Electrostatic wattmeter shunted. The electromagnetic wattmeter. Electromagnetic wattmeter, with mutual inductance. Watt-hour meters. Reisz's method of power measurement. The three voltmeter method. The three ammeter method. Transformer methods. Resonance methods. References.

The measurement of power.

WE have seen in Chapter VI that if W be the mean power in watts expended in an alternating current circuit, V and A the effective volts and amperes, and $\cos\alpha$ the power factor, then

$$W = VA\cos\alpha.$$

The maximum possible value of the power is VA, and it has this value only when the phase difference α is zero, and this can only occur when the ratio of the instantaneous volts to the instantaneous amperes $\left(\frac{e}{i}\right)$ is always constant (Chapter VI). If, as is generally the case, α is not zero, then in order to find W we should need to know the shapes of the volt and ampere curves and their time lag relatively to one another. This could be done by means of an oscillograph, as it is not difficult to find the mean value of ei, that is W, from the curves. In practice, however, this is best done by means of some form of wattmeter, or by some of the methods described below. Before describing the electrostatic wattmeter, we will give the theory of the quadrant electrometer as the principle of the two instruments is the same.

The quadrant electrometer.

In the quadrant electrometer, invented by Lord Kelvin, use is made of electrostatic attractions and repulsions in order to measure potential differences. In the ordinary form of this instrument there is an aluminium disc shaped like the figure **8** placed inside an insulated cylindrical metal box which is completely divided into four quadrants (Fig. 57). This flat disc 3 is suspended by a torsion fibre perpendicular to its plane, and its position of equilibrium when the quadrants are at the same potential is shown in Fig. 57.

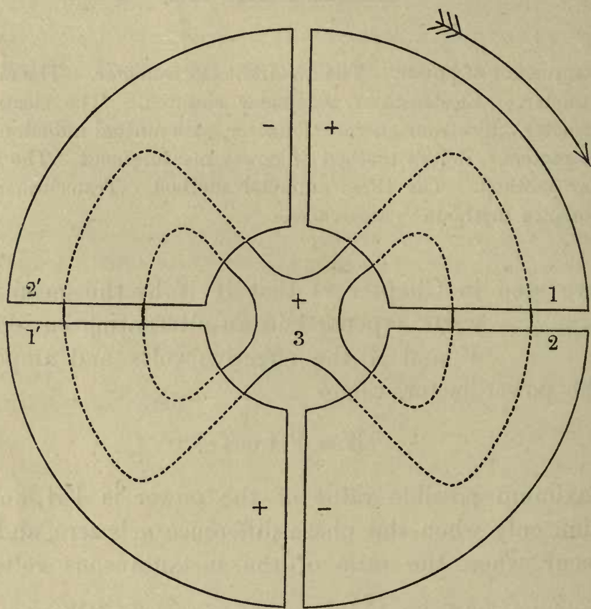

Fig. 57. The Kelvin Quadrant Electrometer.

The opposite quadrants 1 and 1′ are permanently connected by wires, so that they are always at the same potential, and so also are the quadrants 2 and 2′.

Let the potential of the quadrants 1 and 1′ be V_1, of the quadrants 2 and 2′ be V_2 and of the disc be V_3. There will evidently be repulsion between 3 and 1 and attraction between 3 and 2, hence there will be a torque in the direction of the arrowhead. Now if Q_1, Q_2, Q_3 and Q_0 be respectively the quantities of electricity on the two pairs of quadrants, on the disc and on the inside of the earthed screen, which in the case of an electrostatic wattmeter is a metal cover practically enclosing the quadrants, then

$$Q_1 + Q_2 + Q_3 + Q_0 = 0 \quad \text{.....................(1)}.$$

By Chapter IV, we have, since V_0 is zero

$$Q_0 = K_{0.1}V_1 + K_{0.2}V_2 + K_{0.3}V_3,$$

$$Q_1 = K_{1.1}V_1 + K_{1.2}V_2 + K_{1.3}V_3,$$

$$Q_2 = K_{2.1}V_1 + K_{2.2}V_2 + K_{2.3}V_3,$$

$$Q_3 = K_{3.1}V_1 + K_{3.2}V_2 + K_{3.3}V_3.$$

Now (1) must be satisfied for all values of V_1, V_2 and V_3.

Therefore $$K_{0.1} + K_{1.1} + K_{2.1} + K_{3.1} = 0 \quad \text{...............(2)}$$
and $$K_{0.2} + K_{1.2} + K_{2.2} + K_{3.2} = 0 \quad \text{...............(3)}.$$

Now let the disc turn through an angle θ, and let $K'_{1.1}, K'_{1.2}, \ldots$ be the new values of the coefficients. These coefficients must still satisfy equations (2) and (3). Now, owing to the shape of the disc, we see, since the gaps between the quadrants are very narrow, that so long as the edges of 1 and 3 or 1′ and 3′ are not brought too close together

$$K'_{3.3} = K_{3.3} = \text{constant},$$

for the motion merely brings a different part of 3 opposite the gaps between 1 and 2 and between 1′ and 2′. Also since the motion of the disc does not appreciably alter the coefficients of mutual induction for electrostatic charges between 1 and 2, 0 and 1, and 0 and 2, we have

$$K'_{1.2} = K_{1.2}, \quad K'_{0.1} = K_{0.1}, \quad K'_{0.2} = K_{0.2}.$$

Hence from (2)

$$K'_{1.1} + K'_{1.3} = -K'_{0.1} - K'_{1.2}$$

$$= \text{constant}$$

$$= K_{1.1} + K_{1.3} \quad \text{.....................(4)}.$$

Similarly $$K'_{2.2} + K'_{2.3} = K_{2.2} + K_{2.3} \quad \text{.....................(5)}.$$

As θ increases, the parts of the quadrants 1 and 1′ opposed to the disc 3 diminish uniformly, and hence we may write

$$K'_{1.1} = K_{1.1} - \lambda\theta \quad \text{.....................(6)}$$

where λ is a constant. For the same reason

$$K'_{2.2} = K_{2.2} + \lambda\theta \quad \text{.......................(7).}$$

From (4) and (6) $\quad K'_{1.3} = K_{1.3} + \lambda\theta,$

and from (5) and (7) $\quad K'_{2.3} = K_{2.3} - \lambda\theta.$

The electrical energy W of the system in the new position is given by (see p. 90)

$$W = \tfrac{1}{2} K'_{1.1} V_1^2 + \tfrac{1}{2} K'_{2.2} V_2^2 + \tfrac{1}{2} K'_{3.3} V_3^2 + K'_{2.3} V_2 V_3 + K'_{3.1} V_3 V_1 + K'_{1.2} V_1 V_2.$$

Now the work done by the electrical forces on the disc when it turns through an angle $d\theta$

$= \text{The moment of the forces about its axis} \times d\theta,$ but, when potentials are kept constant, the gain dW in the electrical energy (see p 400) is equal to the work done on the disk, is therefore given by ~~therefore~~ $dW = \text{Torque} \times d\theta.$

Hence, $\quad \text{Torque} = \dfrac{dW}{d\theta}.$

If therefore V_1, V_2 and V_3 are kept constant, we see that the torque equals

$$\tfrac{1}{2} V_1^2 \frac{dK'_{1.1}}{d\theta} + \tfrac{1}{2} V_2^2 \frac{dK'_{2.2}}{d\theta} + V_2 V_3 \frac{dK'_{2.3}}{d\theta} + V_3 V_1 \frac{dK'_{3.1}}{d\theta}$$

$$= -\tfrac{1}{2}\lambda V_1^2 + \tfrac{1}{2}\lambda V_2^2 - \lambda V_2 V_3 + \lambda V_3 V_1$$

$$= \lambda (V_1 - V_2)\{V_3 - \tfrac{1}{2}(V_1 + V_2)\}.$$

If the needle be put in metallic connection with the quadrants, 1 and 1′, then V_3 equals V_1 and the formula becomes

$$\text{Torque} = \frac{\lambda}{2}(V_1 - V_2)^2.$$

Used in this manner the electrometer becomes a voltmeter. Equilibrium is attained when the torsional couple is equal to the electrical couple. With direct currents the reading is proportional to the square of the P.D. between its terminals, and with alternating currents it is proportional to the mean value of the square of the P.D. The most satisfactory way of reading the deflection is by means of a ray of light reflected on to a scale from a light mirror fixed to the axis of the disc. If the scale is direct reading, then the divisions on the upper part of the scale will be much larger than those lower down as the deflections increase as the square of the applied voltage.

This instrument, shown diagrammatically in Fig. 58, is practically a slight modification of the quadrant electrometer described above.

The electrostatic wattmeter.

In the figure, Q_1 represents one pair of the quadrants, Q_2 another pair, and N the needle or disc. The disc is generally

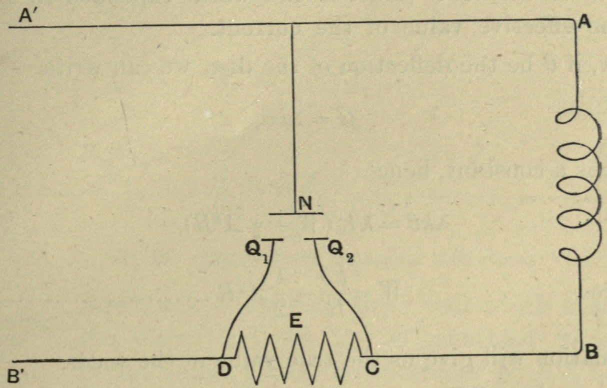

Fig. 58. Connections of the Electrostatic Wattmeter.

connected to an insulated terminal on the case of the instrument by means of a phosphor bronze suspension strip which replaces the torsion fibre in the electrometer. Suppose that we wish to measure the electric power being expended in the coil AB. Place a small resistance $R\,(DC)$ in the main BB' and connect the disc N of the instrument to A. Let v_1 be the instantaneous value of the potential of A, v_2 the potential of B, and v_3 the potential of D. Then, by what we have shown above, the torque g acting on the disc will be given by

$$g = \lambda\,(v_2 - v_3)\,\{v_1 - \tfrac{1}{2}(v_2 + v_3)\},$$

where λ is a constant depending on the instrument.

If DC be non-inductive, then $v_2 - v_3 = Ri$, where i is the instantaneous value of the current. Hence

$$g = \lambda Ri\,\{v_1 - \tfrac{1}{2}(v_2 + v_3)\}.$$

Now, if E be the middle point of DC, its potential is $\frac{1}{2}(v_2 + v_3)$

and $v_1 - \frac{1}{2}(v_2 + v_3)$ is the P.D. between A and E. Therefore, if G be the mean value of the torque g,

$$G = \lambda R \times \text{watts expended in } AE$$

$$= \lambda R \left(W + \tfrac{1}{2} A^2 R\right),$$

where W is the true value of the watts expended in AB and A is the effective value of the current.

Now, if θ be the deflection of the disc, we can write

$$G = \lambda k \theta,$$

where k is a constant, hence

$$\lambda k \theta = \lambda R \left(W + \tfrac{1}{2} A^2 R\right),$$

therefore $$W = \frac{k\theta}{R} - \frac{1}{2} A^2 R \qquad \text{.......................(1).}$$

This equation will give us the true value of the watts.

If we replace R by another resistance R', we have

$$W = k\frac{\theta}{R'} - \frac{1}{2} A^2 R'.$$

Thus we can increase the range of the loads that the wattmeter can measure by making a suitable set of resistances to put in series with the mains. In practice it is not desirable that the potential drop at full load across the series resistance should be more than one per cent. of the potential difference between the mains.

The constant k can be found by using the instrument to measure the power expended in a non-inductive circuit. For example, if V be the effective voltage across AB and θ_1 be the deflection in this case, then

$$k = \frac{R}{\theta_1}\left(VA + \tfrac{1}{2} A^2 R\right).$$

In practice it is not convenient to have a P.D. of more than about 200 volts between the disc and the quadrants. Hence for measuring power in high tension circuits the instrument has to be used with a shunt.

The connections for this case are shown in Fig. 59. $A'B'$

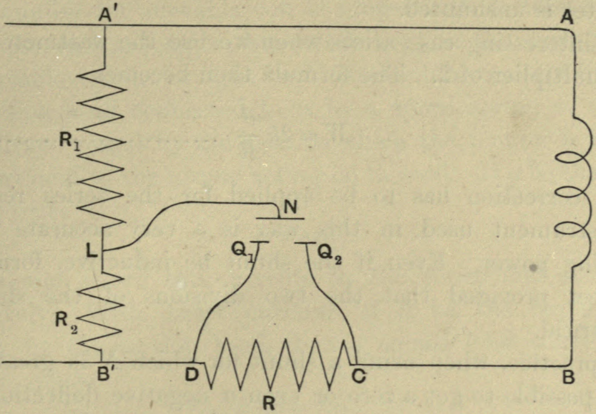

Fig. 59. Electrostatic Wattmeter with shunt.

Electrostatic wattmeter shunted.

represents a non-inductive shunt which is put across the mains and the disc is connected to a point L in the shunt. Let the resistance of $A'L$ be R_1 and of LB' be R_2. Let v_1 and v_1' be the potentials of A' and L respectively, then since the shunt is non-inductive,

$$\frac{v_1 - v_3}{R_1 + R_2} = \frac{v_1' - v_3}{R_2},$$

thus

$$v_1' - v_3 = \frac{1}{N}(v_1 - v_3),$$

where

$$N = \frac{R_1 + R_2}{R_2}$$

= the multiplying power of the shunt.

As formerly,

$$\begin{aligned} g &= \lambda (v_2 - v_3)\{v_1' - \tfrac{1}{2}(v_2 + v_3)\} \\ &= \lambda Ri\{v_1' - v_3 - \tfrac{1}{2}(v_2 - v_3)\} \\ &= \lambda Ri\left\{\frac{1}{N}(v_1 - v_3) - \frac{1}{2}(v_2 - v_3)\right\} \\ &= \frac{\lambda R}{N}(w + Ri^2) - \frac{1}{2}\lambda R^2 i^2. \end{aligned}$$

Thus

$$G = \frac{\lambda R}{N}(W + RA^2) - \frac{1}{2}\lambda R^2 A^2,$$

and

$$W + RA^2 = \frac{N}{\lambda R}\left(\lambda k\theta + \frac{1}{2}\lambda R^2 A^2\right),$$

therefore

$$W = Nk\frac{\theta}{R} + \frac{N-2}{2}RA^2 \quad \text{..................(2).}$$

In this formula for W, k has the same value as when the wattmeter is unshunted.

An interesting case arises when we use the wattmeter with a shunt multiplier of 2. The formula then becomes

$$W = 2k\frac{\theta}{R} \quad \text{..........................(3)}$$

and no correction has to be applied for the series resistance. The instrument used in this way is a very accurate one for measuring power. Even if the shunt be inductive, formula (3) is correct provided that the two divisions of the shunt are symmetrical.

In practice, when using a shunt for which N is greater than 2, it is possible to get a zero or even a negative deflection of the instrument with low power factors. Mr Addenbrooke, who first called the author's attention to this interesting fact, showed him that an electrostatic wattmeter with a shunt for which N is equal to 10 gave a large negative deflection when measuring the power absorbed by a condenser load. It is easy to see that when W is less than $\frac{N-2}{2}RA^2$ we must get a negative reading on the instrument.

When the deflection is zero, then

$$W = \frac{N-2}{2}RA^2 \quad \text{........................(4).}$$

Hence the instrument could be used to measure power by means of a null method. R_2 could easily be made variable so that it could be adjusted until the deflection were zero. Then since $N = \frac{R_1 + R_2}{R_2}$, W could be found from (4) and the reading of an ammeter in the circuit.

The electromagnetic wattmeter.

In the general form of electromagnetic wattmeter we have two coils, of which one is fixed and carries the main current while the other, which is movable and in series with a high nearly non-inductive resistance, is placed as a shunt across the circuit. The coils are placed with their axes at right angles, so that when currents are passing through both there is a force tending to turn them so as to make their axes coincide. The torque on the movable coil is proportional to the product of the two currents so long as that coil is kept in the same position. The movable coil is brought back to its initial position by means of a torsion head connected to it by a spiral spring so that the angle turned round by the head to which the pointer is attached is proportional to the torque acting on the coil. If g be the torque, we may write

$$g = \lambda i i_1,$$

where i and i_1 are the currents in the series and shunt coils respectively. With direct currents, if θ be the angle turned through by the torsion head, then g equals $k'\lambda\theta$, where k' is a constant, hence

$$k'\lambda\theta = \lambda i i_1.$$

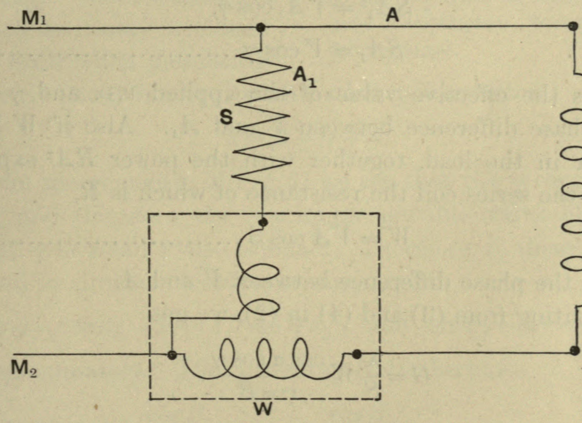

Fig. 60. Electromagnetic Wattmeter.

Now if S be the total resistance of the shunt circuit (Fig. 60) and e the P.D. applied at its terminals,

$$k'\lambda\theta = \lambda i \frac{e}{S}.$$

Therefore $$\theta = \frac{1}{Sk'} W,$$

and thus $$W = k\theta \quad\text{.................................(1)}$$

where k equals Sk' and is constant. The watts then will be proportional to the reading of the instrument, and, since in this case they are equal to the product of the volts and the amperes, k can easily be determined.

With alternating currents we still have

$$g = \lambda i i_1,$$

where g, i and i_1 are the instantaneous values of the torque and amperes respectively. Therefore the mean value of g (G) is given by

$$G = \lambda \times \text{the mean value of } ii_1$$
$$= \lambda A A_1 \cos\alpha \quad\text{.................................(2)}$$

where α is the phase difference between the periodic functions i and i_1 and A and A_1 are their R.M.S. values. Now if there is no iron in the shunt coil, no mutual inductance between the coils, and the eddy currents in the instrument itself are negligible, we have

$$SA_1^2 = VA_1 \cos\gamma,$$

and thus $$SA_1 = V\cos\gamma \quad\text{...........................(3)}$$

where V is the effective value of the applied P.D. and γ is the angle of phase difference between V and A_1. Also if W be the true power in the load, together with the power RA^2 expended in heating the series coil the resistance of which is R,

$$W = VA\cos\beta \quad\text{...........................(4)}$$

where β is the phase difference between V and A.

Substituting from (3) and (4) in (2), we get

$$G = \frac{\lambda}{S} W \frac{\cos\alpha\cos\gamma}{\cos\beta}.$$

Now G equals $\lambda k'\theta$, where θ is the mean value of the deflection. With the frequencies used in practice, θ is constant, hence finally

$$W = k\theta \times \frac{\cos\beta}{\cos\alpha\cos\gamma} \quad\text{.........................(5).}$$

Now the wattmeter is calibrated with direct currents, and therefore indicates $k\theta$ watts. We see that, when the wattmeter is used on an alternating current circuit, the readings of the instrument must be multiplied by

$$\frac{\cos\beta}{\cos\alpha\cos\gamma}.$$

In Fig. 61, where V, A and A_1 are represented graphically, it must be remembered that we cannot assume that their vectors are in one plane. In general therefore β will be less than $\alpha+\gamma$. If there were no phase difference between the applied P.D. and the current in the shunt coil, then γ would be zero, and β would be equal to α. In this case the instrument would indicate alternating current power correctly. In practice however, γ, although small, cannot be neglected, as it may introduce a large error when measuring the electric power in circuits with a low power factor.

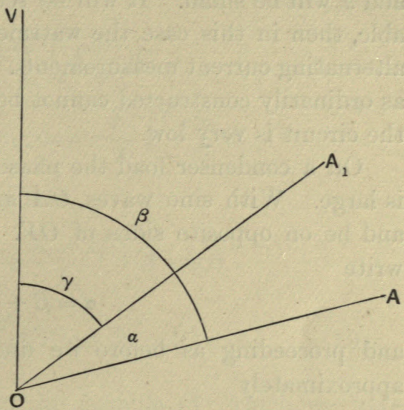

Fig. 61. Currents in shunt and series coils of wattmeter. Vectors are in different planes.

Let $$\alpha=\beta-\gamma+x,$$

then from the geometry of Fig. 61 we see that $\beta+\gamma$ cannot be less than α, and therefore the maximum possible value of x is 2γ. Hence if γ is small x also is small. Therefore, if these angles be expressed in circular measure, we may write

$$\cos\gamma=\cos(\gamma-x)=1,\quad \sin\gamma=\gamma \quad\text{and}\quad \sin(\gamma-x)=\gamma-x$$

very approximately. The correcting factor becomes

$$\frac{\cos\beta}{\cos\alpha\cos\gamma}=\frac{\cos\beta}{\cos\{\beta-(\gamma-x)\}}$$
$$=\frac{\cos\beta}{\cos\beta+(\gamma-x)\sin\beta}\quad\ldots\ldots\ldots\ldots(6).$$

For very low power factors, when β is nearly ninety degrees, (6) may have almost any value, as it depends chiefly on the value of $\gamma-x$. If the potential difference and current waves are sine shaped, OV, OA and OA_1 will be in the same plane, and if the inductance of the shunt coil be greater than its capacity, as it generally is in practice, OA and OA_1 will lie on the same side of OV for inductive loads and x will be zero. If the waves are approximately sine shaped, the vectors will be nearly in one plane and x will be small. It will be seen from (6) that if γ is appreciable, then in this case the wattmeter generally reads too high in alternating current measurements. The electromagnetic wattmeter as ordinarily constructed cannot be used when the power factor of the circuit is very low.

On a condenser load the phase difference α between A and A_1 is large. With sine waves, OA and OA_1 would lie in one plane and be on opposite sides of OV. In this case we may therefore write

$$\alpha = \beta + \gamma - x,$$

and proceeding as before we find that the correcting factor is approximately

$$\frac{\cos\beta}{\cos\beta - (\gamma - x)\sin\beta} \quad \text{.....................(7).}$$

When $\gamma - x = \cot\beta$ we see that the wattmeter reads zero, and when $\gamma - x$ is greater than $\cot\beta$ we get a negative deflection. Now when $\cos\beta$ is very small so also is $\cot\beta$. The wattmeter therefore does not give trustworthy readings on a condenser load when the power factor is low. The errors of this instrument are as a rule greater the higher the frequency, as γ is practically proportional to the frequency.

Electromagnetic wattmeter with mutual inductance.

In the theory of the electromagnetic wattmeter discussed above we have assumed that the shunt and series coils are so situated that the mutual inductance between them is zero. We shall now consider the case when the mutual inductance can not be neglected. Let the constants of the shunt coil be (S, L_1), of the series coil (R, L) and let M be the mutual inductance between the coils. Then if e and v are the instantaneous values of the potential differences across the shunt coil and across the load respectively, our equations are

$$e = Ri + v + L\frac{di}{dt} + M\frac{di_1}{dt} \quad \text{..................(8)}$$

$$e = Si_1 + L_1\frac{di_1}{dt} + M\frac{di}{dt} \quad \text{.....................(9).}$$

Multiplying both sides of equations (8) and (9) by i and taking mean values we get

$$VA\cos\beta = W + \frac{M}{T}\int_0^T i\frac{di_1}{dt}\,dt$$

and

$$VA\cos\beta = SAA_1\cos\alpha + \frac{L_1}{T}\int_0^T i\frac{di_1}{dt}\,dt.$$

Thus eliminating the integral we have

$$W = \frac{MS}{L_1}AA_1\cos\alpha + \frac{L_1 - M}{L_1}VA\cos\beta$$

$$= \frac{M}{L_1}k\theta + \frac{L_1 - M}{L_1}VA\cos\beta \quad\text{.....................(10).}$$

When M is zero this reduces to the formula (4) which we have discussed above. In the particular case when M equals L_1 the formula (10) becomes

$$W = k\theta,$$

and therefore, when the self inductance of the shunt coil equals the mutual inductance between the coils, the instrument reads correctly.

The above theorem suggests the following method of constructing an electromagnetic wattmeter. Suspend the shunt coil so that its self inductance equals the mutual inductance between the coils, and adjust the controlling spring till the pointer reads zero in this position. When it is used to measure power, and currents are passing through both coils, the shunt coil is brought back to its initial position by means of a torsion head and the pointer reads the power. The scale is evenly divided, and one reading with direct current is sufficient to find the constant of the instrument. In order that M may be equal to L_1 it is necessary that the self inductance of the series coil should be greater than the self inductance of the shunt coil.

Watt-hour meters.

Various kinds of watt-hour meters are used in practice to measure the total energy expended in a given time in an installation. If the meter be theoretically correct, the number of revolutions of the spindle in a given time must be proportional to the watt-seconds consumed. If r be the rate of the meter, then

$$r = \frac{\text{Revolutions}}{\text{Watt-Seconds}}$$

$$= \frac{n}{Wt}.$$

Hence $$W = \frac{n}{rt}.$$

Therefore if r be known W can be found.

Most watt-hour meters cannot, however, be trusted to read accurately when the power factor of the circuit is low unless very special precautions are taken in their manufacture. Consider for example the well-known Elihu Thomson watt-hour meter. In this meter the shunt circuit consists of a drum wound armature, the axis of which is the spindle. The armature is in series with a high nearly non-inductive resistance. This is placed in the field due to the series coils, and rotates for the same reason that the armature of a motor rotates. Now, if A_1 be the current in the shunt circuit, then, since the field due to the series coils is proportional to the current in them, that is to the main current A, we see that the mean value of the driving torque is proportional to $AA_1 \cos \alpha$. The retarding torque when the spindle rotates is due to the eddy currents induced by permanent magnets in a disc of copper fixed to the spindle. Owing to the inductance of the circuits in which the eddy currents flow, they do not come instantaneously into existence, neither do they die away instantaneously. By Lenz's law they retard the motion of the disc and, since the field due to the permanent magnets is constant, their values are proportional to the angular velocity ($2\pi n$) of the spindle. Therefore, neglecting friction, the retarding torque is proportional to n, and the driving torque to $AA_1 \cos \alpha$. Proceeding as in the case of the electromagnetic wattmeter, and assuming that the mutual inductance between the shunt and series circuits is negligible, we find that

$$W = \frac{n}{rt} \frac{\cos \beta}{\cos \alpha \cos \gamma}$$

where r is the rate found by direct currents. Hence we see that unless γ is zero, *i.e.* unless the shunt circuit is absolutely non-inductive, the reading on an inductive load will in general be too high. Also on a condenser load if $\gamma - x$ is greater than $\cot \beta$, the meter will run backwards. In actual meters a few extra windings of the shunt circuit are arranged so as to help the field due to the main current and thus enable the meter to start on a small load. It is found that the friction of the spindle in its bearings is very nearly constant, and, if the extra windings be arranged to compensate for this, then $AA_1 \cos \alpha$ will be very approximately proportional to n.

Reisz's method of power measurement.

For this method we require an electrostatic voltmeter, an ammeter and two high non-inductive resistances of known values R_1 and R_2. Let BAC (Fig. 62) be the circuit in which we wish to measure the power.

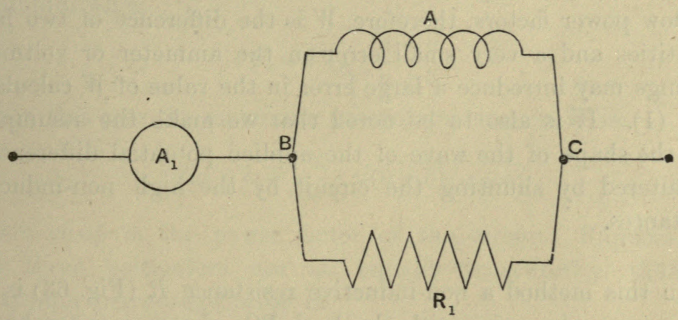

Fig. 62. Reisz's method.

Place an ammeter A_1 in the main circuit, and place R_1 as a shunt across the circuit BAC. Then if i_1 be the main current, e the P.D. across BC and i the current in BAC, we have

$$i_1 = i + \frac{e}{R_1}.$$

Hence
$$i_1^2 = i^2 + \frac{e^2}{R_1^2} + \frac{2w}{R_1},$$

where $w = ei =$ the instantaneous watts expended in BAC.

By summation $$A_1^2 = A^2 + \frac{V^2}{R_1^2} + \frac{2W}{R_1}.$$

Similarly $$A_2^2 = A^2 + \frac{V^2}{R_2^2} + \frac{2W}{R_2}.$$

Therefore $$W = b(A_1^2 - A_2^2) - aV^2 \quad \text{.........................(1)}$$

where $$a = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

and $$b = \frac{R_1 R_2}{2(R_2 - R_1)}.$$

If R_2 is infinite, then (1) becomes

$$W = \frac{R_1}{2}(A_1^2 - A_2^2) - \frac{V^2}{2R_1} \quad \text{......................(2).}$$

If the power factor were zero, W would be zero, and we see from (1) that $b(A_1^2 - A_2^2)$ would then have its minimum value aV^2. For low power factors, therefore, W is the difference of two large quantities and a very small error in the ammeter or voltmeter readings may introduce a large error in the value of W calculated from (1). It is also to be noted that we make the assumption that the shape of the wave of the applied potential difference is not altered by shunting the circuit by the high non-inductive resistances.

The three voltmeter method.

In this method a non-inductive resistance R (Fig. 63) is put in series with the load BC. Let e, e_1 and e_2 be the instantaneous values of the P.D.'s across AC, AB and BC respectively, then if i be the current in the circuit, we shall have at every instant

$$e = e_1 + e_2.$$

Therefore $$e^2 = e_1^2 + e_2^2 + 2e_1 e_2$$
$$= e_1^2 + e_2^2 + 2Rie_2.$$

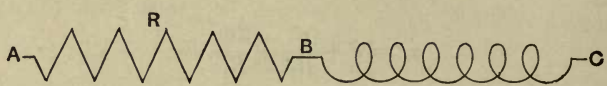

Fig. 63. Three Voltmeter method.

If, therefore, W be the average value of the power ie_2 expended in BC we have by summation

$$V^2 = V_1^2 + V_2^2 + 2RW,$$

and thus
$$W = \frac{1}{2R}(V^2 - V_1^2 - V_2^2)$$

$$= \frac{A}{2V_1}(V^2 - V_1^2 - V_2^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where A is the effective value of the current measured by an ammeter and V, V_1 and V_2 are the readings of voltmeters placed across AC, AB and BC respectively. If the supply voltage is sufficiently steady, one voltmeter can be used to read V, V_1 and V_2.

Suppose that three voltmeters are used and that $\pm a$, $\pm b$, $\pm c$ are the percentage errors of the readings V, V_1 and V_2 and that the ammeter reads correctly, then it can be shown that the maximum possible error in the value of W given by (1) is a minimum when

$$V_2 = V_1 \sqrt{\frac{a+c}{a+b}},$$

and in this case the maximum percentage error in W is

$$2a + b + \frac{2}{\cos\phi}\sqrt{(a+b)(a+c)}$$

where $\cos\phi$ is the power factor of the circuit. Suppose that the three voltmeters are all equally trustworthy, then the maximum percentage error is

$$3a + \frac{4a}{\cos\phi}.$$

If the voltmeters could be trusted to read correctly to within $\pm$ 0·2 per cent., then for a power factor of unity the maximum error would be 1·4 per cent. and for a power factor of 0·01 it would be 80·6 per cent. Owing to the unavoidable errors of observation, with ordinary commercial instruments and on circuits where the voltage is unsteady, formula (1) often gives negative values for W when $\cos\phi$ is small.

The practical objection to this method is that the testing pressure applied between A and C has to be nearly double the working pressure applied between B and C, and this increased pressure is not always available. Another difficulty is that the shape of the wave of the applied potential difference is generally altered when we put a non-inductive resistance in series with the load. The first of these difficulties can be met by using the three ammeter method.

The three ammeter method.

In this method a non-inductive resistance R (Fig. 64) is placed as a shunt across the load, and ammeters are placed in the branch circuits and in the main. If i be the instantaneous value of the current in the main, then

$$\dot{i} = \dot{i}_1 + \dot{i}_2.$$

Therefore

$$\dot{i}^2 = \dot{i}_1^2 + \dot{i}_2^2 + 2\dot{i}_1\dot{i}_2$$

$$= \dot{i}_1^2 + \dot{i}_2^2 + \frac{2}{R} e\dot{i}_2,$$

where e is the applied P.D. Hence by summation

$$A^2 = A_1^2 + A_2^2 + \frac{2}{R} W.$$

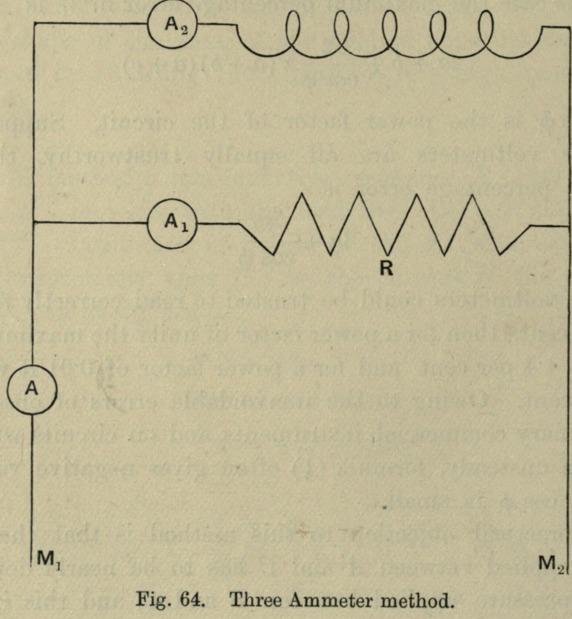

Fig. 64. Three Ammeter method.

Thus
$$W = \frac{R}{2}(A^2 - A_1^2 - A_2^2)$$
$$= \frac{V}{2A_1}(A^2 - A_1^2 - A_2^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where V is the applied P.D. and A_1 the current in the non-inductive resistance. When the power factor is very small, W is small and hence, from (1), A^2 is nearly equal to $A_1^2 + A_2^2$. Therefore in this case a small error in measuring any of the currents may introduce a large error into the value of W calculated from (1).

Transformer methods. There are several methods of measuring power by means of transformers. These are in general based on the fact that the phase difference between the primary and secondary voltage of a transformer on a light load is almost exactly 180 degrees. Let e_1 and e_2 be the primary and secondary voltages, i_1 and i_2, r_1 and r_2, n_1 and n_2 be the currents, resistances and turns respectively; then if Φ be the flux in the core and there is no magnetic leakage,

$$e_1 = r_1 \dot{i}_1 + n_1 \frac{d\Phi}{dt},$$

$$e_2 + r_2 \dot{i}_2 = -n_2 \frac{d\Phi}{dt},$$

therefore
$$e_1 + \frac{n_1}{n_2} e_2 = r_1 \dot{i}_1 - \frac{n_1}{n_2} r_2 \dot{i}_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, in a modern closed iron circuit commercial transformer on open secondary, the maximum value of e_1 is about ten thousand times greater than the maximum value of $r_1 i_1$ and therefore, in this case, we can write

$$e_2 = -\frac{n_2}{n_1} e_1.$$

This equation shows that e_2 and e_1 are similar curves and differ in phase by 180 degrees.

The method shown in Fig. 65 is due to Albert Campbell. A small non-inductive resistance R is placed in series with the load, and the primary of a suitable transformer is placed across it. By means of a reversing switch S the secondary voltage e_2 of this transformer can be added or subtracted from the P.D. e applied across the load, and the resultant measured by an electrostatic voltmeter. Let v_1 and v_2 be the two resultant voltages, then

$$v_1 = e + e_2$$

$$= e - \frac{n_2}{n_1} e_1$$

$$= e - \frac{n_2}{n_1} Ri',$$

where i' is the current in R.

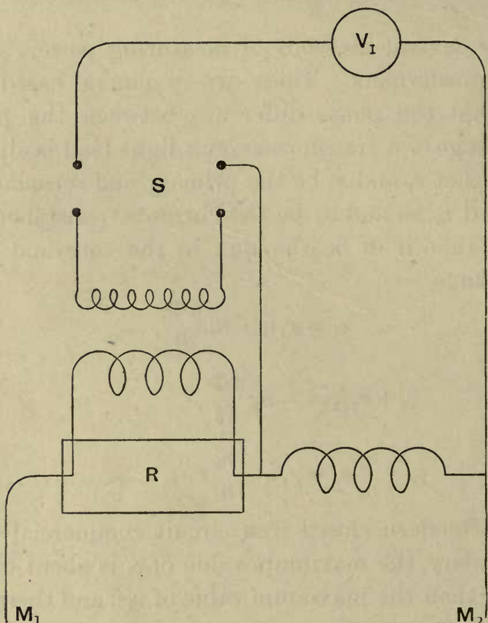

Fig. 65. Campbell's method.

Now, if the magnetising current of the transformer be very small compared with the current i in the load, then i' is approximately equal to i and

$$v_1 = e - \frac{n_2}{n_1} Ri.$$

Thus $$V_1^2 = V^2 + \frac{n_2^2}{n_1^2} R^2 A^2 - 2\frac{n_2}{n_1} RW.$$

Similarly $$V_2^2 = V^2 + \frac{n_2^2}{n_1^2} R^2 A^2 + 2\frac{n_2}{n_1} RW.$$

Hence $$W = \frac{n_1}{n_2} \frac{V_2^2 - V_1^2}{4R} \quad \text{.........(1)}$$
$$= k(V_2 - V_1)(V_2 + V_1) \quad \text{.........(2)}$$
where k is a constant.

This method, like the three voltmeter method, fails when W is very small; but as a series of values of V_2 and V_1 can rapidly be taken it is practically convenient in other cases.

Elihu Thomson, Aron and other watt-hour meters for high tension circuits are often provided with a transformer for the volt coil and make use of the fact that for light loads the primary and secondary volt waves are in opposition in phase.

The series coils of meters also are sometimes connected across the secondaries of transformers the primaries of which are in series with high tension mains. In this case the magnetising current of the transformer must be small, as it is assumed that the primary and secondary currents are in a constant ratio to one another and that they are in opposition in phase.

Resonance methods.

For measuring the dielectric losses in condensers, Rosa and Smith make use of the principle of resonance. A drum of flexible cable AD (Fig. 66) is put in series with the condenser K, which, for example, may be a concentric or a polyphase cable on open circuit. A and B are connected to low pressure supply mains, and a wattmeter W is placed in the circuit so that it measures the power taken by both the inductive coil and the condenser. If the core of the cable be made of many strands of very fine wire and there be no iron or metal near and the frequency be not too high, then the power consumed by the coil will be very nearly A^2R where A is the current in the circuit and R is the resistance of the coil. Now if the inductance of the coil be adjusted by unwinding some of the cable, then, for a certain inductance, resonance of the first harmonic will ensue and the voltage V across the condenser may be 20 or 30 times greater than the P.D. applied across AB. In this case the power factor of the resonant circuit may be nearly unity and the wattmeter will read the load W accurately. Therefore, subtracting the power expended in the inductive coil, we find $W - A^2R$ as the condenser loss. In this case, although the power factor may be as small as 0·01, the loss could be measured with fair accuracy.

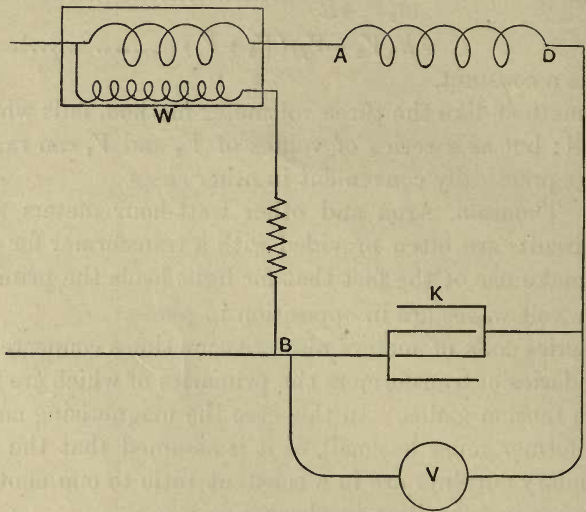

Fig. 66. Resonance method.

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CHAPTER X.

The air core transformer. Equivalent net-work. Variable secondary load. Secondary load inductive. Condenser in secondary load. Circle diagram for a leaky transformer. Transformer with no leakage. Maxwell's formula. Rimington's theorem. Leading primary current with condenser load. Transformer with constant primary current.

The air core transformer.

IF we have two neighbouring electric circuits and an alternating current be flowing in one of them, then, owing to their mutual induction, an alternating E.M.F. will be induced in the other, and if this be a closed circuit an alternating current will be set up in it. The magnitude of this induced current will depend on the relative position of the two circuits and the permeability of the medium in which they are placed. We shall for the present consider that there are no magnetic materials near the circuits, so that we may take the permeability of the medium to be unity; the two circuits will then have a constant mutual inductance. This is the problem of the air-core transformer and, as its solution is of fundamental importance in alternating current theory, we will first of all attempt to solve it without making any assumption as to the shape of the wave of the applied P.D.

Equivalent net-work.

Let e be the instantaneous value of the P.D. applied to the primary circuit, whose resistance and inductance are R ohms and L henrys respectively, and let S and N be the resistance and inductance of the secondary circuit, and M the mutual inductance of the two circuits. If i_1 and i_2 be the instantaneous values of the currents in the primary and secondary circuits we have

$$\left.\begin{aligned} e = Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt} \quad\dots\dots\dots\dots(1),\\ 0 = Si_2 + N\frac{di_2}{dt} + M\frac{di_1}{dt} \quad\dots\dots\dots\dots(2).\end{aligned}\right\}$$

These equations may be written in the forms

$$\left.\begin{aligned} e = Ri_1 + L\frac{dI}{dt} \quad\dots\dots\dots\dots(3),\\ -M\frac{dI}{dt} = Si_2 + N\sigma\frac{di_2}{dt} \quad\dots\dots\dots\dots(4),\end{aligned}\right\}$$

where $$I = i_1 + \frac{M}{L}i_2$$

and $$\sigma = 1 - \frac{M^2}{LN}.$$

The quantity σ is called the leakage factor of the transformer. We shall see in Volume II that a knowledge of its value is of fundamental importance in the theory of transformers.

Eliminating $\frac{dI}{dt}$ between (3) and (4) we get

$$-\frac{M}{L}(e - Ri_1) = Si_2 + N\sigma\frac{di_2}{dt} \quad\dots\dots\dots\dots(5),$$

and thus $$e - Ri_1 = \frac{L^2}{M^2}Si' + \frac{L^2}{M^2}N\sigma\frac{di'}{dt} \quad\dots\dots\dots\dots(6),$$

where $$i' = -\frac{M}{L}i_2 \quad\dots\dots\dots\dots(7).$$

Hence finally

$$\left.\begin{aligned} i_1 = I + i' \quad\dots\dots\dots\dots(8),\\ e - Ri_1 = \frac{L^2}{M^2}Si' + \frac{L^2}{M^2}N\sigma\frac{di'}{dt} \quad\dots\dots\dots\dots(9),\\ e - Ri_1 = L\frac{dI}{dt} \quad\dots\dots\dots\dots(10).\end{aligned}\right\}$$

These three equations (8) (9) and (10) show us that the problem of finding i_1 is the same as that of finding the current in the coil AB in Fig. 67, where the resistance of AB is R, the resistance and inductance of BCE are $\frac{L^2}{M^2}S$ and $\frac{L^2}{M^2}N\sigma$ respectively, and the inductance of the choking coil BDE is L.

The solution of the transformer problem is therefore identical with the solution of the problem of finding the currents in the comparatively simple (seeing that there is no mutual induction) net-work shown in Fig. 67, where the primary P.D. (e) is supposed to be applied between A and E. By (7) we see that the current in the secondary circuit is in exact opposition in phase to the

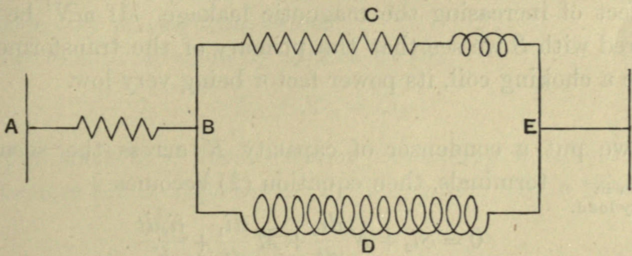

Fig. 67. Equivalent net-work of a transformer. Resistance of AB is R; resistance and inductance of BCE are $\frac{L^2}{M^2}S$ and $\frac{L^2}{M^2}N\sigma$ which is equal to $\frac{L^2}{M^2}N\left(1-\frac{M^2}{LN}\right)$. Inductance of BDE is L. Current in the primary is the same as the current in AB.

current in BCE and its magnitude is $\frac{L}{M}$ times this current. Again, if there is a non-inductive resistance across the secondary terminals, the P.D. between the terminals will be in phase with the secondary current and therefore in opposition to the phase of the current in BCE.

Variable secondary load.

This construction still applies when S is a variable resistance, as for example a spark gap. Hence we should expect that when a transformer is supplying an electric arc, the peculiar shape of the P.D. wave at the terminals of the arc would be transmitted through the transformer into the primary circuit. This has been proved experimentally by Duddell and Marchant.

Again, if we put an inductive load on the secondary, then, from equation (2), we alter the value of N and therefore also the value of σ. Suppose that the inductance of the secondary load is N'. Then the inductance of the imaginary coil BCE (Fig. 67) is

Secondary load inductive.

$$\frac{L^2}{M^2}(N+N')\,\sigma' = \frac{L^2}{M^2}\left(N+N'-\frac{M^2}{L}\right) = \frac{L^2}{M^2}(N\sigma + N').$$

Therefore the leakage factor becomes $\sigma + \frac{N'}{N}$. Thus the effect of adding inductance to the secondary is exactly the same as the effect of increasing the magnetic leakage. If $\omega N'$ be great compared with S we see that the primary of the transformer will act like a choking coil, its power factor being very low.

If we put a condenser of capacity K across the secondary terminals, then equation (2) becomes

Condenser in secondary load.

$$0 = Si_2 + N\frac{di_2}{dt} + M\frac{di_1}{dt} + \frac{\int i_2 dt}{K}.$$

Hence from (6)

$$e - Ri_1 = \frac{L^2}{M^2}Si' + \frac{L^2}{M^2}N\sigma\frac{di'}{dt} + \frac{L^2}{M^2}\frac{\int i' dt}{K}.$$

Therefore to get the diagram for this case we have to put a condenser of capacity $\frac{M^2}{L^2}K$ in series with an inductive coil of resistance $\frac{L^2}{M^2}S$ and inductance $\frac{L^2}{M^2}N\sigma$.

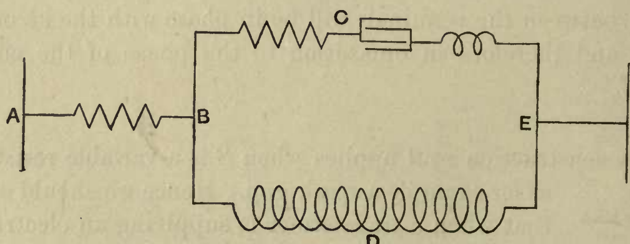

Fig. 68. Equivalent net-work of transformer with a condenser load. The circuit BCE has resistance $\frac{L^2}{M^2}S$, inductance $\frac{L^2}{M^2}N\sigma$ and capacity $\frac{M^2}{L^2}K$. Inductance of BDE equals L. Current in the primary equals the current in AB.

When studying formulae connected with transformers it will often be found convenient to employ diagrams similar to Figs. 67 and 68. Then, merely by inspection, we can see what the effect, for example, will be of diminishing the leakage factor or putting inductance in the secondary circuit.

Circle diagram for a leaky transformer.

In the particular case when the resistance R of the primary circuit is zero, we see that the current I in the choking coil BDE in the equivalent net-work is always the same whatever may be the load on the secondary of the transformer.

If we denote its value by i_0, then by (8)

$$Li_1 + Mi_2 = Li_0.$$

Therefore there is a linear relation connecting i_1, i_2 and i_0, and their vectors can be represented by lines which form a triangle.

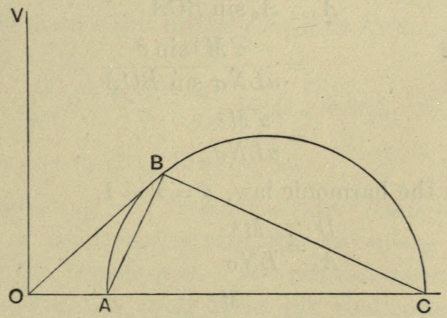

Fig. 69. The circle ABC is the locus of the extremity of the primary current vector of a leaky transformer on certain assumptions. $AC = A_0 \dfrac{1-\sigma}{\sigma}$.

In Fig. 69 let OA, OB and BA represent the effective values of i_0, i_1 and $\dfrac{M}{L} i_2$ in magnitude and phase respectively, and let OV represent V the P.D. across the primary terminals. OV will be at right angles to OA but it will only be in the same plane as the triangle OAB in particular cases. We will however make this assumption in order to simplify the diagram. Draw BC perpendicular to AB. Now if θ be the angle between OV and AB, then

$$\sin\theta = \frac{\alpha\omega \frac{L^2}{M^2} N\sigma A'}{V}$$

$$= \frac{\alpha\omega \frac{L}{M} N\sigma A_2}{V},$$

where $\frac{\omega}{2\pi}$ is the frequency and α is a quantity that depends on the shape of the current wave in the secondary, but cannot be less than unity.

Similarly $$V = \alpha'\omega L A_0.$$

Now if we describe a circle on AC as diameter, it will pass through B. If AC equals D, then by trigonometry

$$\frac{D}{A_0} = \frac{\frac{M}{L} A_2}{A_0 \sin BCA}$$

$$= \frac{\alpha' M^2 \sin\theta}{\alpha L N\sigma \sin BCA}$$

$$= \frac{\alpha' M^2}{\alpha L N\sigma}.$$

If e follows the harmonic law, $\alpha = \alpha' = 1$,

thus $$\frac{D}{A_0} = \frac{M^2}{LN\sigma}$$

$$= \frac{M^2}{LN - M^2}$$

$$= \frac{1-\sigma}{\sigma}.$$

In the elementary theory of the induction motor this ratio is important.

Transformer with no leakage.

In the particular case when there is no leakage, σ equals zero and D is infinite. It follows that AB is perpendicular to OA. This can also be seen at once, as Fig. 67 in this case simplifies down to a perfect choking coil shunted by a non-inductive resistance.

Maxwell's formula.

When the applied P.D. wave is sine-shaped, we can apply the method of the complex variable (Chapter VII) to solve the problem corresponding to Fig. 67. It easily follows that the effective resistance of the primary is $R+m^2S$ and its inductance is $L-m^2N$ where m^2 is $\frac{M^2\omega^2}{S^2+N^2\omega^2}$. These results were proved by Clerk Maxwell in his paper on 'A dynamical theory of the electromagnetic field,' published in the *Philosophical Transactions* for 1865, p. 475.

Rimington's theorem.

If the magnetic leakage be negligible, then in the equivalent net-work (Fig. 67) we have a non-inductive resistance R in series with a choking coil of inductance L shunted by a non-inductive resistance $\frac{L^2}{M^2}S$. In this case it is known (Chapter VII) that there is a certain value of S which will make the current in R a minimum. Hence putting a non-inductive load on the secondary sometimes makes the primary current smaller. From page 168 we see that the particular value of S that makes the primary current a minimum is given by the equation

$$\frac{L^2}{M^2}S=\frac{\omega^2L^2}{2R}+\frac{\omega L}{2R}\{\omega^2L^2+4R^2\}^{\frac{1}{2}}.$$

This theorem is due to E. C. Rimington.

Leading primary current with condenser load.

A most interesting and important case arises when we put a condenser across the secondary terminals of the transformer. We can see from the diagram (Fig. 68) that, when the condenser is small, it will supply some of the current required to magnetise the choking coil, and hence the primary current will be diminished. As we increase the capacity of the condenser the primary current attains a minimum value and then begins to increase again. When the primary current is a minimum, the phase difference between it and the applied P.D. is small, but it is only zero when the P.D. wave is sine-shaped. As we increase the capacity the angle of lead of the current attains a maximum value and then begins to diminish again. Finally, since a very large condenser acts simply like a short circuit, the current in the primary ends by lagging nearly ninety degrees in phase behind the P.D. If there were no magnetic leakage, it would be possible to obtain leading primary currents of any magnitude.

Transformer with constant primary current.

If the resistance of the primary circuit be negligible and A is the effective value of the primary current, it easily follows, by applying the method of the complex variable to Fig. 68, that

$$\frac{A^2}{V^2} = \frac{1}{L^2\omega^2} - \frac{M^2}{L^2} \frac{2 - KN(1+\sigma)\omega^2}{KL\omega^2 \left\{ S^2 + \left(N\sigma - \frac{1}{K\omega^2} \right)^2 \omega^2 \right\}}.$$

If we make
$$K = \frac{2}{N(1+\sigma)\omega^2},$$
then
$$A = \frac{V}{L\omega},$$
whatever be the value of S.

If ϕ be the angle of phase difference between the primary current and the applied P.D., then we have in this case

$$\tan\phi = \frac{4\frac{L^2}{M^2}S^2 - \omega^2 M^2}{4SL\omega}.$$

Hence as S diminishes from infinity to zero, the lag of the primary current changes from $+90$ degrees to -90 degrees, although the current itself remains constant all the time.

Such an air core transformer would be useful as a phase regulator for testing whether a wattmeter reads correctly on loads of various power factors. On open circuit it would read nearly zero; as we diminished the resistance in the secondary circuit, the readings would increase to a maximum value AV and then gradually diminish, attaining a minimum value when the resistance in the secondary was zero. We should have to make sure however that the harmonics in the applied P.D. wave were negligible.

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CHAPTER XI.

Three phase alternators. The magnitudes and the phase differences of the voltages between the mains in three phase systems. The graphical representation of the voltages across the three arms of the load (star connected). Rule for finding the voltages across a star load when the resistances of the arms are given. Algebraical formula for finding the angles of the voltage triangle. Example. The voltages in a star load adjust themselves so that the power expended in it is a minimum. Algebraical relations between the various voltages. The potentials to earth of the mains. To find the potential of the centre of a star load which is insulated from earth. The capacity currents in the sheath. Diagram of the currents in a mesh load. Algebraical formulae for the currents in a mesh load. The form of the wave of P.D. in three phase systems. Examples. The P.D. waves between the mains can only be similar curves when their effective values are all equal. To find the frequency of the alternating current in the fourth wire of a balanced three phase system. Value of load on a three phase alternator at every instant. The measurement of power in three phase circuits. Watt-hour meters. Minimum value of the sum of the three voltages in a star load. To find the resistances of the arms of a star load in order to get symmetrical three phase currents. To find the ratio of the resistances in a star load in order that the voltages to the centre may be equal. References.

Three phase alternators.

THE fundamental principle of the commonest type of three phase alternator is illustrated in Fig. 70. Insulated wire is wound evenly round a laminated iron ring and connections are made on three points on the wire at equal angular distances apart, with the three terminals of the machine. A powerful magnet *NS* rotates inside this ring, and as the magnetic circuit is completed through the two halves of the ring it will be seen that the flux inside the coils will change in direction every half revolution, and hence an E.M.F.

will be set up in each coil. If everything is symmetrical, the effective P.D.'s between any two of the three terminals must be equal. The ring represents the armature of the three phase alternator and the magnet the revolving field magnets. In this case the armature is said to be mesh wound and the three windings are in series with one another. Any want of balance amongst the E.M.F.'s generated in the three windings of the

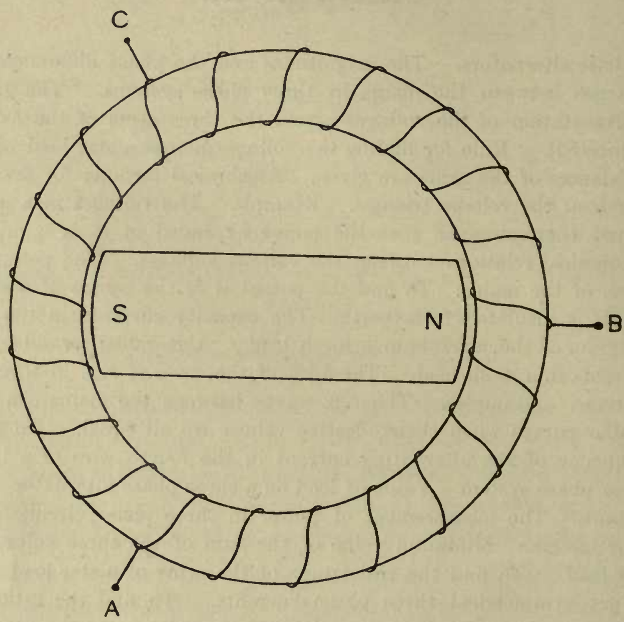

Fig. 70. Three phase machine with mesh-connected armature.

armature will cause alternating currents to flow in the windings even at no load, and will thus lower the efficiency of the machine. It is therefore important to make the armature as symmetrical as possible.

The mains are connected with the three terminals of the machine and the load can be placed across two of the mains or divided symmetrically between the three of them either in mesh fashion like the windings of the armature or in star fashion like the armature shown in Fig. 71.

The principle of three phase motors can be understood from Fig. 70. If we were to connect A, B and C to three phase supply mains then currents would flow round the three windings of the armature, and we shall see in Chapter XIV that a rotating magnetic field will be produced which will cause the magnet to make f revolutions per second, where f is the frequency of the alternating current.

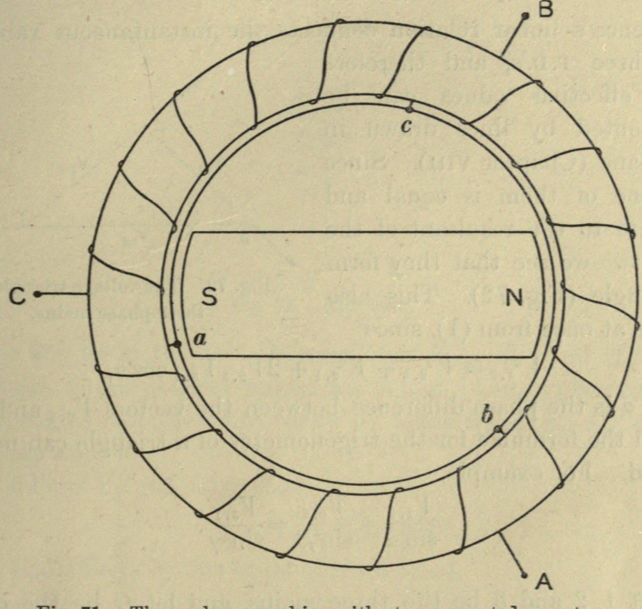

Fig. 71. Three phase machine with star-connected armature.

A diagrammatic sketch of a three phase alternator with its armature star wound is shown in Fig. 71. A, B and C are the terminals of the machine, and a, b and c, the other ends of the three windings, are joined together. In this case there is obviously no current flowing in the coils when the machine is on open circuit.

When the three P.D.'s across the terminals of a three phase alternator are all equal then the load is said to be balanced. In practice it is not always possible to obtain a balance, and so we will first consider the relations that hold between the effective P.D.'s and their phase differences in the general case.

Let v_1 be the potential of No. 1 main, v_2 the potential of No. 2 main and v_3 the potential of No. 3 main.

The magnitudes and the phase differences of the voltages between the mains in three phase systems.

Let also

$v_{1.2} = v_1 - v_2 =$ the P.D. between mains 1 and 2.

$v_{2.3} = v_2 - v_3 =$ the P.D. between mains 2 and 3.

$v_{3.1} = v_3 - v_1 =$ the P.D. between mains 3 and 1.

Then obviously $v_{1.2} + v_{2.3} + v_{3.1} = 0$ ....(1).

Hence a linear relation connects the instantaneous values of the three P.D.'s, and therefore their effective values can be represented by lines drawn in one plane (Chapter VIII). Since any one of them is equal and opposite to the resultant of the other two we see that they form a triangle (Fig. 72). This also follows at once from (1), since

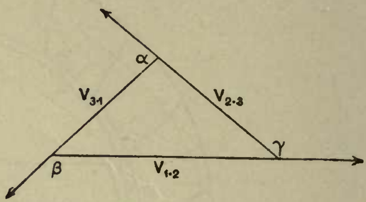

Fig. 72. The voltage triangle for three phase mains.

$$V^2_{1.2} = V^2_{2.3} + V^2_{3.1} + 2V_{2.3}V_{3.1}\cos\alpha,$$

where α is the phase difference between the vectors $V_{2.3}$ and $V_{3.1}$.

All the formulae for the trigonometry of a triangle can now be applied. For example,

$$\frac{V_{1.2}}{\sin\alpha} = \frac{V_{2.3}}{\sin\beta} = \frac{V_{3.1}}{\sin\gamma}.$$

The graphical representation of the voltages across the three arms of the load (star connected).

Let 1, 2 and 3 be the three mains, and let O be the centre of the star load. If p, q and r be the resistances of the three arms (non-inductive) and e_1, e_2 and e_3 be the P.D.'s across the arms, then, if O is earthed and the capacity currents in the sheath can be neglected, since by Kirchhoff's law the algebraical sum of the instantaneous values of the currents at O must be zero, we have

$$\frac{e_1}{p} + \frac{e_2}{q} + \frac{e_3}{r} = 0.$$

Hence, as before,

$$\frac{E_1^2}{p^2} = \frac{E_2^2}{q^2} + \frac{E_3^2}{r^2} + 2\frac{E_2E_3}{qr}\cos\theta_{2.3} \text{...............(1)}$$

and two similar equations, where E_1 is the voltage across the arm $1O$, and $\theta_{2.3}$ is the angle of phase difference between e_2 and e_3.

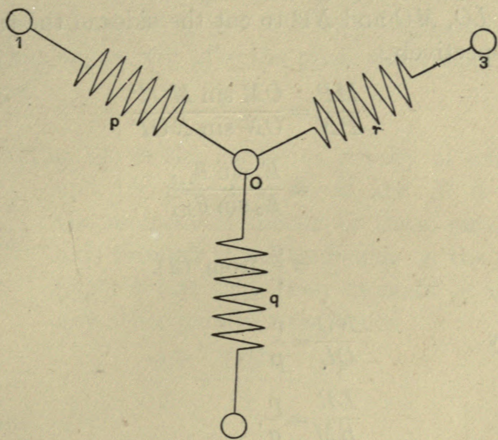

Fig. 73. Star load.

We see, as in the last theorem, that E_1/p, E_2/q and E_3/r form a triangle whose exterior angles are $\theta_{2.3}$, $\theta_{3.1}$ and $\theta_{1.2}$ respectively. These angles can therefore easily be found when E_1, E_2, E_3, p, q and r are known.

It follows by geometry that

$$\theta_{1.2}+\theta_{2.3}+\theta_{3.1}=360^\circ \quad \text{.....................(2).}$$

It also follows by the 'rule of sines' that

$$\frac{E_1}{p\sin\theta_{2.3}}=\frac{E_2}{q\sin\theta_{3.1}}=\frac{E_3}{r\sin\theta_{1.2}} \quad \text{...............(3).}$$

From any point O (Fig. 74) draw OL equal to E_1. Make the angle LON equal to $\theta_{3.1}$ and make ON equal to E_3. Make the angle NOM equal to $\theta_{2.3}$ and OM equal to E_2. Then by (2) the angle LOM equals $\theta_{1.2}$.

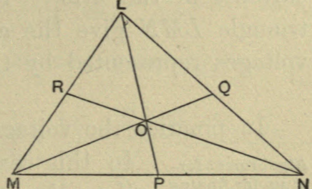

Fig. 74. O is the C. of G. of masses $\frac{1}{p}$, $\frac{1}{q}$ and $\frac{1}{r}$ at L, M and N respectively.

Now,

$$v_{1.2}=v_1-v_2=e_1-e_2,$$

$$\therefore V^2_{1.2}=E_1^2+E_2^2-2E_1E_2\cos\theta_{1.2} \quad (4)$$

$$=LM^2 \text{ in Fig. 74.}$$

Therefore LM equals $V_{1.2}$. Similarly, MN and NL equal $V_{2.3}$ and $V_{3.1}$ respectively. Hence the triangle LMN is the voltage triangle already found (Fig. 72).

Produce LO, MO and NO to cut the sides of the triangle in P, Q and R respectively.

Then
$$\frac{MP}{PN} = \frac{OM \sin MOP}{ON \sin NOP}$$
$$= \frac{E_2 \sin \theta_{1.2}}{E_3 \sin \theta_{3.1}}$$
$$= \frac{q}{r} \text{ from (3).}$$

Similarly
$$\frac{NQ}{QL} = \frac{r}{p},$$

and
$$\frac{LR}{RM} = \frac{p}{q}.$$

It follows from these relations that O is the centre of gravity of three particles of masses $1/p$, $1/q$ and $1/r$ placed at L, M and N respectively.

Rule for finding the voltages across a star load when the resistances of the arms are given.

Construct a triangle LMN the lengths of whose sides represent the voltages between the mains. Find the centre of gravity O of masses $1/p$, $1/q$ and $1/r$ placed at L, M and N. Then OL equals E_1, the voltage across the terminals of the resistance p (Fig. 73), OM equals E_2, and ON equals E_3. The angles LOM, MON and NOL give the angles of phase difference between the various voltages and also, since the arms are non-inductive, between the currents in the arms. The supplements of the angles of the triangle LMN give the angles of phase difference between the voltages represented by the sides of the triangle.

Algebraical formula for finding the angles of the voltage triangle.

In practice the voltages between the mains are nearly equal. In this case, if the voltages be V, $V + xV$ and $V + yV$, we can find the angles of the triangle in degrees by the approximate formulae

$$60 - 33(x + y); \quad 60 + 33(2y - x); \quad 60 + 33(2x - y).$$

Suppose that the P.D.'s between the mains are 2000, 2060 and 2140 volts respectively. By the formulae the angles of the voltage triangle are 56·7, 59·7 and 63·6 degrees. Their true values are 56·6, 59·6 and 63·8 degrees. The supplements of these angles will give the phase differences between the three voltages.

Example.

Since O (Fig. 74) is the centre of gravity of masses $1/p$, $1/q$ and $1/r$ placed at L, M and N, it follows that the moment of inertia of these masses about an axis through O perpendicular to the plane of the paper is less than their moment of inertia about any other parallel axis, hence

The voltages in a star load adjust themselves so that the power expended in it is a minimum.

$$\frac{OL^2}{p} + \frac{OM^2}{q} + \frac{ON^2}{r}$$

is a minimum.

Since p, q and r are non-inductive, this expression represents the power being expended in the star load.

Since
$$\frac{e_1}{p} + \frac{e_2}{q} + \frac{e_3}{r} = 0,$$

Algebraical relations between the various voltages.

we have
$$e_1\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = \frac{v_{1.2}}{q} - \frac{v_{3.1}}{r}.$$

Hence, squaring and taking mean values,

$$E_1^2\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)^2 = \frac{V^2_{1.2}}{q^2} + \frac{V^2_{3.1}}{r^2} - \frac{2V_{1.2}V_{3.1}\cos\beta}{qr}.$$

But $2V_{1.2}V_{3.1}\cos\beta = V^2_{2.3} - V^2_{3.1} - V^2_{1.2}$.

Hence
$$E_1^2\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)^2 = \left(\frac{V^2_{1.2}}{q} + \frac{V^2_{3.1}}{r}\right)\left(\frac{1}{q} + \frac{1}{r}\right) - \frac{V^2_{2.3}}{qr} \quad \text{...(5)}.$$

We can write down two similar equations by symmetry. Similarly

$$\frac{V^2_{1.2}}{pq} = \left(\frac{E_1^2}{p} + \frac{E_2^2}{q}\right)\left(\frac{1}{p} + \frac{1}{q}\right) - \frac{E_3^2}{r^2} \quad \text{.........(6)}$$

and two other similar equations. Hence, also,

$$\left(\frac{E_1^2}{p} + \frac{E_2^2}{q} + \frac{E_3^2}{r}\right)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = \frac{V^2_{1.2}}{pq} + \frac{V^2_{2.3}}{qr} + \frac{V^2_{3.1}}{rp} \quad \text{...(7)}.$$

It follows that if p, q and r are all equal, then

$$V^2_{1.2} + V^2_{2.3} + V^2_{3.1} = 3\,(E_1^2 + E_2^2 + E_3^2) \quad\text{.........(8).}$$

If in addition $V_{1.2} = V_{2.3} = V_{3.1}$,

then $V_{1.2} = \sqrt{3}E_1$ ....(9).

This can also easily be proved from Fig. 74, since in this case LMN is an equilateral triangle. It is to be noted that no assumption is made as to the shape of the P.D. waves.

The potentials to earth of the mains.

Let v_1, v_2 and v_3 be the potentials of the three mains, and let f_1, f_2 and f_3 be their fault resistances to earth. By the fault resistance of a main we mean the resistance of all those leakage paths from it to earth which do not pass through the other mains. Then since if the capacity currents in the sheath can be neglected, the sum of the leakage currents to earth must be zero,

$$\frac{v_1}{f_1} + \frac{v_2}{f_2} + \frac{v_3}{f_3} = 0.$$

Proceeding as before, we see that if O is the centre of gravity of masses $1/f_1$, $1/f_2$ and $1/f_3$ placed at the angles L, M, N of the voltage triangle, then OL, OM and ON represent V_1, V_2 and V_3 respectively in magnitude and phase.

It follows also that, if the fault resistances of the mains vary, then their potentials adjust themselves so that the power expended in leakage currents is a minimum. An analogous theorem holds true for the case of three wire direct current systems.

To find the potential of the centre of a star load which is insulated from earth.

Let p, q and r be the resistances of the arms of the star load, and let x be the potential of the centre. Then

$$\frac{v_1 - x}{p} + \frac{v_2 - x}{q} + \frac{v_3 - x}{r} = 0,$$

therefore $$x\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = \frac{v_1}{p} + \frac{v_2}{q} + \frac{v_3}{r}.$$

Now as the P.D. waves have different shapes, a little consideration will show that the frequency of the alternating potential is an unknown quantity.

If V denote the effective value of x, then

$$V^2\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)^2=\frac{V_1^2}{p^2}+\ldots+2\,\frac{V_2V_3}{qr}\cos\phi_{2.3}+\ldots\ldots\ldots(1).$$

Also, if f_1, f_2 and f_3 be the fault resistances of the mains to earth, and if the capacity currents in the sheath can be neglected, then

$$\frac{V_1^2}{f_1^2}=\frac{V_2^2}{f_2^2}+\frac{V_3^2}{f_3^2}+2\,\frac{V_2V_3}{f_2f_3}\cos\phi_{2.3}$$

and two similar equations.

By means of these equations we can eliminate the cosines from (1) and we get finally

$$V^2\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)^2=\frac{V_1^2}{f_1^2}\left(\frac{f_1}{p}-\frac{f_2}{q}\right)\left(\frac{f_1}{p}-\frac{f_3}{r}\right)$$
$$+\frac{V_2^2}{f_2^2}\left(\frac{f_2}{q}-\frac{f_3}{r}\right)\left(\frac{f_2}{q}-\frac{f_1}{p}\right)$$
$$+\frac{V_3^2}{f_3^2}\left(\frac{f_3}{r}-\frac{f_1}{p}\right)\left(\frac{f_3}{r}-\frac{f_2}{q}\right).$$

Hence if
$$\frac{f_1}{p}=\frac{f_2}{q}=\frac{f_3}{r},$$
we see that V is zero and x also is zero.

The capacity currents in the sheath.

Let us now consider the case of a three core cable with a lead sheath. In practice the sheath is earthed and so, approximately at least, it is at zero potential. We saw in Chapter IV that, so far as capacity effects are concerned, we can, in theoretical diagrams, replace a three core cable by three small conductors joined by six condensers (see Fig. 39). The effects of leakage can be shown on this diagram by joining $S\,1$, $S\,2$ and $S\,3$ with non-inductive resistances f_1, f_2 and f_3. Since by Kirchhoff's law, the sum of the currents flowing to and from the sheath must be zero, we have

$$\frac{v_1}{f_1}+\frac{v_2}{f_2}+\frac{v_3}{f_3}+K_{1.0}\frac{dv_1}{dt}+K_{2.0}\frac{dv_2}{dt}+K_{3.0}\frac{dv_3}{dt}=0,$$

and thus
$$\frac{v_1}{f_1}+\frac{v_2}{f_2}+\frac{v_3}{f_3}+\frac{dq_0}{dt}=0,$$

where q_0 is the total charge on the sheath at the time t. We may call $\frac{dq_0}{dt}$ the capacity current in the sheath. It is to be noted that it is the sum of a number of small leakage currents taking place

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along paths which may be at considerable distances from each other and therefore the value of the capacity current in the sheath may vary largely at different points along its length. A formula for the effective value C_0 of $\frac{dq_0}{dt}$ can be found as follows. We have

$$-\frac{dq_0}{dt}=\frac{v_1}{f_1}+\frac{v_2}{f_2}+\frac{v_3}{f_3},$$

thus
$$C_0^2=\frac{V_1^2}{f_1^2}+\ldots+2\frac{V_2V_3}{f_2f_3}\cos\phi_{2.3}+\ldots,$$

and therefore
$$C_0^2=\frac{V_1^2}{f_1^2}+\ldots+\frac{V_2^2+V_3^2-V_{2.3}^2}{f_2f_3}+\ldots.$$

Therefore, if we know the star and mesh voltages and the fault resistances of the mains, we can calculate C_0.

Diagram of the currents in a mesh load.

Let a_1, a_2 and a_3 be the instantaneous values of the currents in the mains and let i_1, i_2 and i_3 be the currents in the mesh windings. The arrowheads are drawn pointing round the same way so as to get algebraical symmetry in our equations. At any instant however either one or two of the quantities i_1, i_2 and i_3 must be negative.

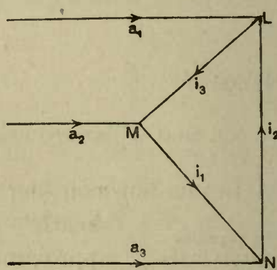

Fig. 75. Currents in a mesh-connected load.

From the figure we see that

$$a_1=i_3-i_2,\quad a_2=i_1-i_3$$

and
$$a_3=i_2-i_1\quad\ldots\ldots\ldots(1).$$

Hence,
$$a_1+a_2+a_3=0.$$

Thus, proceeding as before, we see that A_1, A_2 and A_3 form a triangle (Fig. 76), the exterior angles of which give the phase differences between the three vectors.

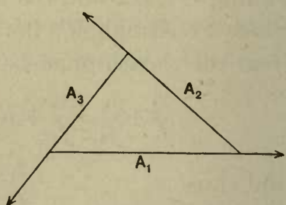

Fig. 76. The phase differences of the currents in the mains.

Again, if r_1, r_2 and r_3 be the resistances of the arms MN, NL and LM in Fig. 75, and if they are non-inductive,

$$r_1i_1=v_2-v_3,$$
$$r_2i_2=v_3-v_1,$$
$$r_3i_3=v_1-v_2.$$

Therefore
$$r_1i_1+r_2i_2+r_3i_3=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We have supposed the three arms of the load non-inductive, but equation (2) is also true in other cases. Suppose, for example, that the three arms of the load are the primary coils of a three phase transformer (see Volume II), each coil being wound with n turns of wire, then

$$r_1 \dot{i}_1 + n\frac{d\phi_1}{dt} = v_2 - v_3, \text{ etc.},$$

where ϕ_1 is the magnetic flux in the first limb of the transformer. Thus

$$r_1\dot{i}_1 + r_2\dot{i}_2 + r_3\dot{i}_3 + n\frac{d}{dt}(\phi_1 + \phi_2 + \phi_3) = 0.$$

If there is no magnetic leakage, $\phi_1 + \phi_2 + \phi_3$ must equal zero, since the lines of induction are closed curves, and therefore, as before,

$$r_1\dot{i}_1 + r_2\dot{i}_2 + r_3\dot{i}_3 = 0.$$

Hence as formerly

$$\frac{r_1 I_1}{\sin\theta_{2.3}} = \frac{r_2 I_2}{\sin\theta_{3.1}} = \frac{r_3 I_3}{\sin\theta_{1.2}} \qquad \text{.................(3)},$$

where $\theta_{2.3}$ is the phase difference between I_2 and I_3. If $r_1 I_1$, $r_2 I_2$ and $r_3 I_3$ represent forces acting at a point O, equations (3) show that they will be in equilibrium.

Now it is known from statics that O is the centre of gravity of three equal masses placed at the extremities of $r_1 I_1$, $r_2 I_2$ and $r_3 I_3$ respectively. Hence O will also be the centre of gravity of masses proportional to r_1, r_2 and r_3 placed at the extremities of lines equal in length to I_1, I_2 and I_3 respectively.

Now from (1)

$$A_1^2 = I_3^2 + I_2^2 - 2I_3 I_2 \cos\theta_{2.3}.$$

If we draw therefore from O (Fig. 77) lines proportional to I_1, I_2 and I_3 and inclined to one another at angles $\theta_{2.3}$, $\theta_{3.1}$ and $\theta_{1.2}$, we see that the lines joining the extremities of these lines give us A_1, A_2 and A_3 respectively. Hence the triangle is the same triangle as in Fig. 76.

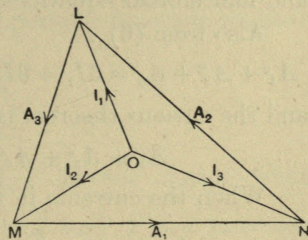

Fig. 77. The currents in the mains and in the arms of the mesh load.

Knowing the currents in the mains and the resistances of the arms of the mesh, we can find the currents in the arms by the following construction.

Make a triangle LMN (see Fig. 77) whose sides are equal to A_1, A_2 and A_3 respectively. Find the centre of gravity O of masses r_1, r_2 and r_3 placed at L, M and N. Then OL will give I_1, OM will give I_2, and ON will give I_3.

Again, if in Fig. 77 we produce OL to L', etc. so that $OL' = r_1 . OL$, $OM' = r_2 . OM$, and $ON' = r_3 . ON$, then if the resistances are non-inductive, $L'M'N'$ is the triangle giving the voltages between the mains and OL', OM' and ON' give the voltages between the mains and the centre of the star load.

Algebraical formulae for the currents in a mesh load.

We can prove the following algebraical relations between the currents in the same way as we proved the relations between the P.D.'s.

$$r_2 r_3 A_1^2 = (r_2 I_2^2 + r_3 I_3^2)(r_2 + r_3) - r_1^2 I_1^2 \quad \text{......(4)}$$

and two similar equations. Similarly,

$$(r_1 + r_2 + r_3)^2 I_1^2 = (r_2 A_3^2 + r_3 A_2^2)(r_2 + r_3) - r_2 r_3 A_1^2 \quad \text{...(5)}$$

and two similar equations. Hence, also

$$(I_1^2 r_1 + I_2^2 r_2 + I_3^2 r_3)(r_1 + r_2 + r_3)$$
$$= A_1^2 r_2 r_3 + A_2^2 r_3 r_1 + A_3^2 r_1 r_2 \quad \text{......(6).}$$

The case when r_1, r_2 and r_3 are all equal is important. O (Fig. 78) is now the centre of gravity of the triangle LMN.

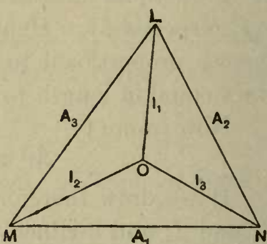

Fig. 78. The relations between the currents.

Hence, from geometry or from equations (4) and (5) above,

$$A_1^2 = 2(I_2^2 + I_3^2) - I_1^2 \quad \text{......(7),}$$

$$9 I_1^2 = 2(A_2^2 + A_3^2) - A_1^2 \quad \text{.....(8),}$$

and four similar equations.

Also from (6)

$$A_1^2 + A_2^2 + A_3^2 = 3I_1^2 + 3I_2^2 + 3I_3^2 \quad \text{...(9),}$$

and the curious theorem is also true that

$$A_1^4 + A_2^4 + A_3^4 = (3I_1^2)^2 + (3I_2^2)^2 + (3I_3^2)^2 \quad \text{.........(10).}$$

When the currents in the mains are all equal,

$$A = \sqrt{3} I \quad \text{..........................(11).}$$

It is to be noted that these formulae are perfectly general, no assumptions having been made as to the shape of the wave.

When we have a star winding, the currents in the branches are of course equal to the currents in the mains. Their phase differences can be got by constructing a triangle whose sides are proportional to the currents. The exterior angles of this triangle are the phase differences required.

If we have both a star and mesh winding, the graphical representation of all the currents by a figure drawn in one plane is rarely possible.

The form of the wave of P.D. in three phase systems.

We will first suppose that the effective P.D.'s between the mains are all equal to one another. Let $Vf(t)$ represent the instantaneous value of one of these P.D.'s, where V is the effective voltage and t is the time in seconds from the era of reckoning. Then the other two will be represented by $Vf\left(t+\frac{T}{3}\right)$ and $Vf\left(t+\frac{2T}{3}\right)$ where T is the period of the alternating current. Since the sum of the instantaneous values of the P.D.'s must always be zero, we must have

$$f(t)+f\left(t+\frac{T}{3}\right)+f\left(t+\frac{2T}{3}\right)=0 \text{..............(1).}$$

This functional equation limits the possible forms of $f(t)$. Solving it by Laplace's method we get

$$f(t)=X\sin\left(\frac{2\pi t}{T}+Y\right),$$

where X and Y are constants or functions of t whose values do not alter when $t+\frac{T}{3}$ or $t+\frac{2T}{3}$ is substituted for t.

Since, however, $f(t)$ is an alternating periodic function, we have another equation to satisfy, namely

$$f(t)=-f\left(t+\frac{T}{2}\right).$$

This adds the condition that X and Y do not change when $t+\frac{T}{2}$ is written for t in them. Hence X and Y are functions of t that do not alter when $t+\frac{T}{3}$, $t+\frac{T}{2}$, or $t+\frac{2T}{3}$ is written for t.

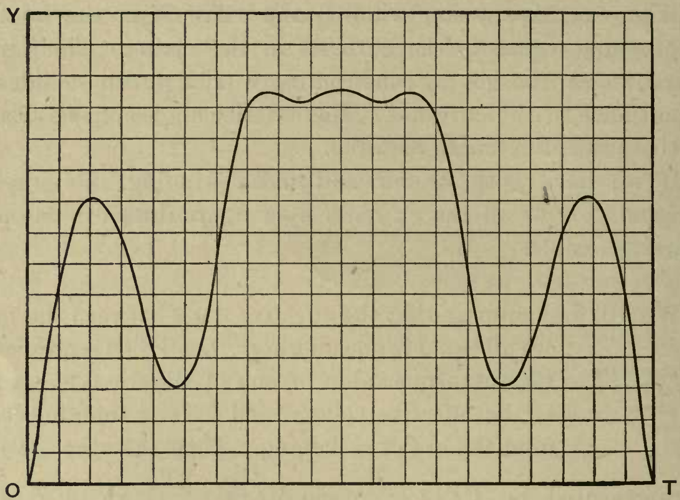

Fig. 79. $y=\sin\left(\frac{2\pi t}{T}+\frac{1}{2}\sin 6\frac{2\pi t}{T}\right)$.

Possible form of P. D. wave in balanced three phase system. Positive half only shown.

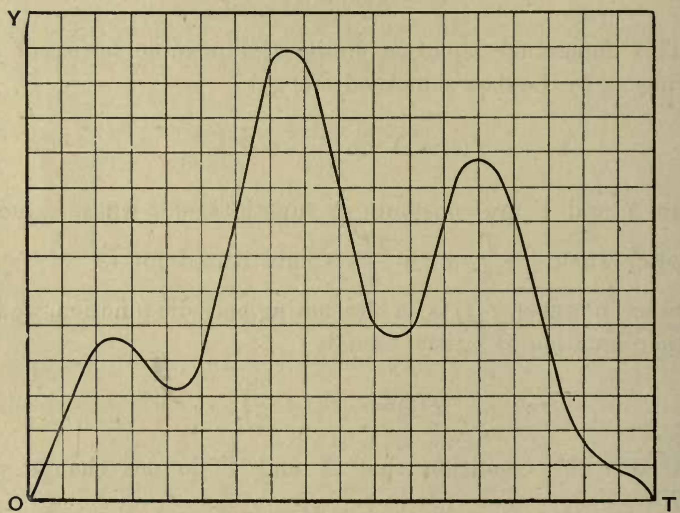

Fig. 80. $y=\left(1+\frac{1}{2}\sin 6\frac{2\pi t}{T}\right)\sin\frac{2\pi t}{T}$.

Another possible form.

In Fig. 79, X has been taken equal to 1, and Y equal to $\frac{1}{2}\sin 6\frac{2\pi t}{T}$, so that the equation to the curve is

Examples.

$$y=\sin\left(\frac{2\pi t}{T}+\frac{1}{2}\sin 6\frac{2\pi t}{T}\right).$$

In Fig. 80 the equation to the curve is

$$y=\left(1+\frac{1}{2}\sin 6\frac{2\pi t}{T}\right)\sin\frac{2\pi t}{T}.$$

The equation (1) can also be solved by Fourier's method. The Fourier series in this case may be written

$$f(t)=\Sigma A_{6n\pm1}\sin\left\{(6n\pm1)\frac{2\pi t}{T}+\alpha_{6n\pm1}\right\} \text{.........(2)}.$$

As a rule it will be found that the expression $X\sin\left(\frac{2\pi t}{T}+Y\right)$ is more convenient to work with than the series given by (2).

The P. D. waves between the mains can only be similar curves when their effective values are all equal.

If it is possible for the potential difference waves between the mains to be similar curves when their effective values are different, let the P.D.'s between the mains be represented by $V_1 f(t)$, $V_2 f\left(t+\frac{T}{3}\right)$ and $V_3 f\left(t+\frac{2T}{3}\right)$ respectively, where V_1, V_2 and V_3 are positive. Then,

at time t, $$V_1 f(t)+V_2 f\left(t+\frac{T}{3}\right)+V_3 f\left(t+\frac{2T}{3}\right)=0,$$

at time $t+\frac{T}{3}$, $$V_3 f(t)+V_1 f\left(t+\frac{T}{3}\right)+V_2 f\left(t+\frac{2T}{3}\right)=0,$$

at time $t+\frac{2T}{3}$, $$V_2 f(t)+V_3 f\left(t+\frac{T}{3}\right)+V_1 f\left(t+\frac{2T}{3}\right)=0.$$

Eliminating the functions by determinants or otherwise, we get

$$(V_1+V_2+V_3)\{(V_2-V_3)^2+(V_3-V_1)^2+(V_1-V_2)^2\}=0,$$

and therefore $$V_1=V_2=V_3,$$

since $V_1+V_2+V_3$ must be positive.

Hence, if we make the assumption that the P.D. waves between the terminals of a three phase alternator are all similar, we also make the assumption that their values are all equal.

By an almost identical proof we can show that the P.D. waves between the mains and earth can only be similar curves when the leakage currents to earth are all equal to one another.

To find the frequency of the alternating current in the fourth wire of a balanced three phase system.

In Fig. 81 let A, B and C be the three terminals of a three phase alternator, the armature of which is star wound. Let also AD, BE and CF be the mains. The points O and L are sometimes put to earth or connected by a wire. We will investigate the frequency of the current in this wire when the loads on the three arms are equal. Let $Cf(t)$, $Cf\left(t+\frac{T}{3}\right)$ and $Cf\left(t+\frac{2T}{3}\right)$

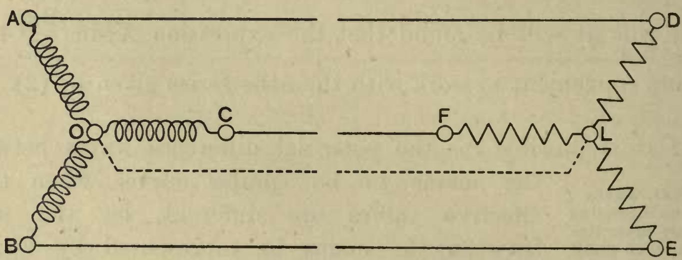

Fig. 81. Frequency of the current in the fourth wire equals $3(2n+1)f$.

be the instantaneous values of the currents in the mains and let $-C'\phi(t)$ be the current in the fourth wire. Then

$$C'\phi(t) = C\left\{f(t) + f\left(t+\frac{T}{3}\right) + f\left(t+\frac{2T}{3}\right)\right\},$$

and
$$C'\phi\left(t+\frac{T}{6}\right) = C\left\{f\left(t+\frac{T}{6}\right) + f\left(t+\frac{T}{2}\right) + f\left(t+\frac{5T}{6}\right)\right\}$$
$$= -C\left\{f\left(t+\frac{2T}{3}\right) + f(t) + f\left(t+\frac{T}{3}\right)\right\}$$
$$= -C'\phi(t),$$

and thus $\phi\left(t+\frac{T}{6}\right) = -\phi(t)$.

Hence $\phi\left(t+\frac{T}{3}\right) = -\phi\left(t+\frac{T}{6}\right) = \phi(t)$.

It follows that, if the frequency of the currents in the mains be f, the frequency of the current in the fourth wire is of the form $3(2n+1)f$, where n is a positive integer. Its lowest value is therefore $3f$.

It is easy to show that, if the current waves in the mains be triangular or parabolic in shape, then the current wave in the fourth wire is also triangular or parabolic, and, since its frequency is three times as great as that of the wave in any main, its effective value is one-third that of the current in each main. If the currents in the mains were sine curves, its value would be zero. Since a sine curve and a parabola differ very little in shape from one another (see Fig. 44), this shows how important a small modification in the shape of the wave may be. In the general case the shape of the resultant wave is quite different from the shape of the component waves.

The above results prove that in practical three phase working slight causes may considerably alter the shape of the E.M.F. and current waves. For example, a slight variation in the resistance or, *a fortiori*, in the inductance of the fourth wire, will alter the shapes of the current waves in the other three wires.

Value of load on a three phase alternator at every instant.

We have shown that, when everything is symmetrical in a three phase system, the potentials of the mains can be expressed by functions of the form $X \sin\left(\frac{2\pi t}{T} + Y\right)$, where X and Y are periodic functions of t whose frequency is $\frac{6}{T}$ or $6f$. If the mesh load between the mains be three non-inductive resistances each equal to R, and the star load arms be each equal to r, then the power at any instant is

$$\frac{1}{R}\left\{X \sin\left(\frac{2\pi t}{T} + Y\right) - X \sin\left(\frac{2\pi t}{T} + Y + \frac{\pi}{3}\right)\right\}^2 + \ldots + \ldots$$

$$+ \frac{1}{r} X^2 \sin^2\left(\frac{2\pi t}{T} + Y\right) + \ldots + \ldots$$

$$= \frac{3}{2}\left(\frac{1}{r} + \frac{1}{R}\right) X^2,$$

where X is a function of t whose frequency is $6f$. Hence the instantaneous value of the power is only constant in special cases.

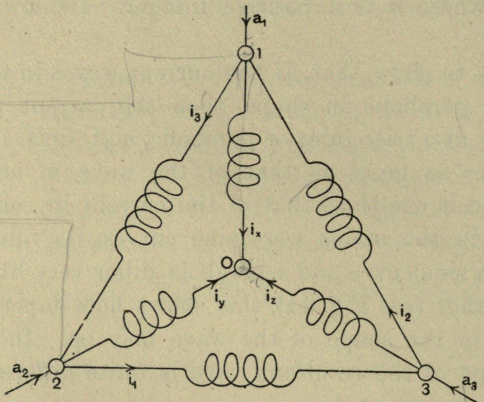

Fig. 82. The measurement of power.

The measurement of power in three phase circuits.

Suppose that there is both a mesh and a star load as in Fig. 82. Let a_1, a_2 and a_3; i_1, i_2 and i_3; i_x, i_y and i_z be the instantaneous values of the currents in the mains, in the arms of the mesh load and in the arms of the star load respectively.

Then
$$a_1 = i_3 - i_2 + i_x,$$
$$a_2 = i_1 - i_3 + i_y,$$
$$a_3 = i_2 - i_1 + i_z.$$

Thus
$$a_1 + a_2 + a_3 = i_x + i_y + i_z = 0,$$
if the three phase machine is insulated from the earth and the leakage currents to the mains are inappreciable.

Let w be the instantaneous value of the watts, then
$$w = v_{1.2} i_3 + v_{2.3} i_1 + v_{3.1} i_2 + v_1 i_x + v_2 i_y + v_3 i_z.$$

Now
$$v_{3.1} = -v_{1.2} - v_{2.3},$$
and
$$i_y = -i_x - i_z.$$
Thus
$$w = v_{1.2}(i_3 - i_2 + i_x) + v_{3.2}(i_2 - i_1 + i_z)$$
$$= v_{1.2} a_1 + v_{3.2} a_3$$
by symmetry
$$= v_{2.3} a_2 + v_{1.3} a_1 \quad \text{.......................(1).}$$
and
$$= v_{3.1} a_3 + v_{2.1} a_2$$

Similarly
$$w = v_1 a_1 + v_2 a_2 + v_3 a_3 \quad \text{.......................(2).}$$

Let us next consider the case when the point O is not maintained at zero potential and let v_x be its potential. Then the power being expended in all the paths of the current a_1 from the point 1 where the potential is v_1 to the point or points where the potential is v_x is $(v_1 - v_x)\,a_1$. Similarly $(v_2 - v_x)\,a_2$ and $(v_3 - v_x)\,a_3$ will be the values of the power in the paths of the two other main currents respectively. Thus the total power w in the load is given by

$$w = (v_1 - v_x)\,a_1 + (v_2 - v_x)\,a_2 + (v_3 - v_x)\,a_3 \text{.........(3)}.$$

If we make the supposition that O is insulated from the earth, then, $a_1 + a_2 + a_3$ must be zero and we get

$$w = v_1 a_1 + v_2 a_2 + v_3 a_3.$$

Similarly $$w = v_{1\,.\,3}\, a_1 + v_{2\,.\,3}\, a_2$$

and two similar equations. Therefore equations (1) and (2) still hold in this case.

When O is connected with the earth by a wire and the leakage currents from the mains are appreciable, or when the centre O' of the star winding of the armature is also connected with the earth, or when O and O' are joined by a fourth wire, then $a_1 + a_2 + a_3$ is not necessarily zero. Let i_0 be the current in the wire joining O to the earth, then we must always have

$$a_1 + a_2 + a_3 + i_0 = 0,$$

and thus by (3)

$$w = v_1 a_1 + v_2 a_2 + v_3 a_3 + v_x i_0.$$

Therefore $v_1 a_1 + v_2 a_2 + v_3 a_3$ is not equal to the power in the load in this case. We also have

$$w = v_{1\,.\,3}\, a_1 + v_{2\,.\,3}\, a_2 - (v_3 - v_x)\, i_0.$$

When i_0 is zero these formulae simplify to (2) and (1) given above.

The formulae (1) and (3) give the methods of measuring power in three phase circuits. The first method is to use two wattmeters. The ampere coil of one of them is put in No. 1 main and the volt coil is connected across 1 and 2. The ampere coil of the other is put in No. 2 main and the volt coil is connected across 3 and 2. Suppose that w_1 is the reading on one meter and that w_2 is the reading on the other, and suppose that w_1 is greater than w_2. Then the power given to the circuit is $w_1 \pm w_2$. If the phase difference between a_3 and $v_{3.2}$ is less than 90 degrees, w_2 is positive, but if greater, w_2 is negative. We must be careful, when measuring, to note whether we have to reverse the shunt connections or not in order that the needle may deflect the right way. If we have to reverse, then w_2 is negative.

The second method is to use three wattmeters, their ampere coils being put in the main circuits and their volt coils across 0 and 1, 0 and 2 and 0 and 3 respectively.

It will be seen that the three wattmeter method is applicable to three phase systems which use a fourth wire. In this case the two wattmeter method cannot be used.

These methods also apply when the load is arranged in any manner between the mains. For example, it may be arranged as in Fig. 83, where 1, 2 and 3 are the terminals for the mains.

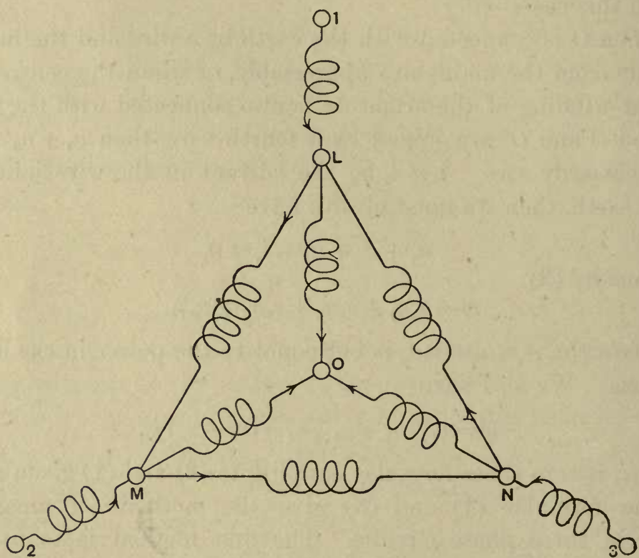

Fig. 83. Possible connections.

An algebraical proof may be given as follows.

Let e_1, e_2 and e_3 denote the P.D.'s between 1 and L, 2 and M and 3 and N respectively, and let O be insulated from earth.

Then $$w = e_1 a_1 + e_2 a_2 + e_3 a_3 + v_{LM} a_1 + v_{NM} a_3$$
$$= (e_1 - e_2 + v_{LM})\, a_1 + (e_3 - e_2 + v_{NM})\, a_3$$
$$= v_{1.2} a_1 + v_{3.2} a_3 \text{ as before.}$$

Similarly $$w = e_1 a_1 + e_2 a_2 + e_3 a_3 + v_{LO} a_1 + v_{MO} a_2 + v_{NO} a_3$$
$$= v_1 a_1 + v_2 a_2 + v_3 a_3.$$

It is to be noted that, if all the mains are equally loaded and the loads are non-inductive, one meter is sufficient. If the volt coil be connected across two of the mains we multiply the reading by 2. If it be connected from one main to the centre of the system, then the multiplying factor is 3. This latter method is also true for balanced inductive loads.

An important case arises when the volt coil of the meter forms one of the arms of a star load, made up of two equal high resistances connected with the volt coil. An arrangement of this kind is generally called a star-box. If the three arms are of equal resistance, the multiplying factor is 3. If, however, as is often the case in practice, the resistance of the volt coil be different from that of the other two arms, then the multiplying factor for balanced loads is $2 + \frac{r}{R}$, where R is the resistance of the volt coil and r that of either of the other arms of the box.

Let O (Fig. 84) be the centre of gravity of masses $1/R$, $1/r$ and $1/r$ placed at A, B and C respectively. Then OA, OB and OC will be the three P.D.'s to the centre of the star-box.

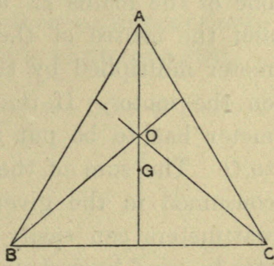

Fig. 84. Voltages in star-box.

Also

$$OA^2\left(\frac{1}{R} + \frac{1}{r} + \frac{1}{r}\right)^2$$
$$= \left(\frac{CA^2}{r} + \frac{AB^2}{r}\right)\left(\frac{1}{r} + \frac{1}{r}\right) - \frac{BC^2}{r}.$$

Thus $$OA^2\left(2 + \frac{r}{R}\right)^2 = 3 \,.\, AB^2 = 9 \,.\, GA^2,$$

where G is the centre of gravity of the triangle ABC;

therefore $$3\,\frac{GA}{OA} = 2 + \frac{r}{R}.$$

Now if the arms had been equal, GA would have been the P.D. across the volt coil. Hence the required multiplying factor is $\frac{3.GA}{OA}$, and this we have shown equals $2+\frac{r}{R}$.

In Fig. 85, G is the generator and M is the motor. If we assume that the three arms are equally loaded, then the energy expended can be measured by means of an ordinary single phase watt-hour meter W and a star-box (S in

Watt-hour meters.

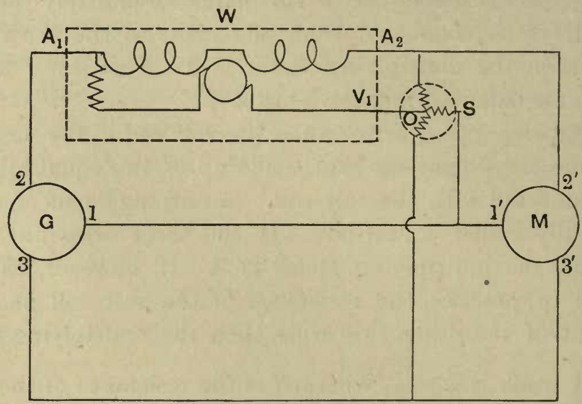

Fig. 85. Connections of watt-hour meter to measure balanced loads.

the figure). The ampere coil A_1A_2 (see Chapter IX) is put in one of the mains 22′ and the volt coil A_1V_1 is connected to 22′ and the centre of the star-box. The units registered by the meter multiplied by three will give the total energy expended on the motor. If the arms are unequally loaded, a watt-hour meter has to be put in each arm and the volt coils connected to O. The sum of the three readings will then give the energy consumed in the given time. It will be seen that the three wattmeters can easily be made into a single one containing a star-box and having six ampere terminals A_1, A_2, etc., three being connected with the supply mains and three being connected with the mains for the load, the volt connections being permanently made inside the meter.

If we connect two single phase watt-hour meters, as in Fig. 86, then by the formulae given above the sum of their readings will give the energy expended. If one of them runs backwards, then its reading has to be subtracted from the reading of the other,

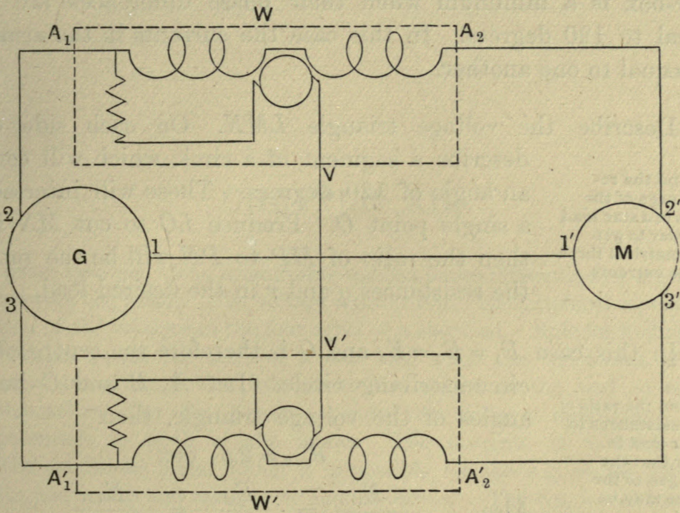

Fig. 86. Connections of Watt-hour Meters to measure all loads. The ampere coils of the two meters are put in two of the mains and the volt coils are connected with the third main.

which will always be the greater. A true three phase meter can be made by combining the two meters (see Chapter XIII). It will have four ampere terminals and one volt terminal. If the load be balanced and non-inductive, the reading on either instrument shown in Fig. 86 multiplied by two will give the true units expended.

Minimum value of the sum of the three voltages in a star load.

It can be shown mathematically that if we take a point O within a triangle LMN (Fig. 87) then

$$OL + OM + ON$$

is a minimum when the angles LOM, MON and NOL are each equal to 120 degrees, provided, of course, that no angle of the triangle is equal to or greater than 120 degrees. If one angle of the triangle is equal to 120 degrees, then, to make $OL + OM + ON$ a mini-

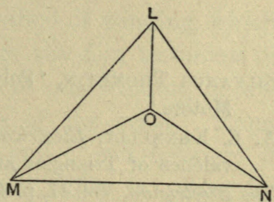

Fig. 87. In a triangle $OL + OM + ON$ is a minimum when the angles at O are all equal.

mum, O must coincide with this angle. It follows therefore from Fig. 74 that the sum of the three voltages across the arms of a star-box is a minimum when their phase differences are each equal to 120 degrees. In this case the currents in the arms are all equal to one another.

To find the resistances of the arms of a star load in order to get symmetrical three phase currents.

Describe the voltage triangle LMN. On each side of it describe a segment of a circle which will contain an angle of 120 degrees. These will intersect in a single point O. Produce LO to cut MN in P, then the ratio of MP to PN will be the ratio of the resistances q and r in the desired load.

To find the ratio of the resistances in a star load in order that the voltages to the centre may be equal.

In this case $E_1 = E_2 = E_3$ and O is therefore the centre of the circumscribing circle. Let A, B and C be the angles of the voltage triangle, then

$$\theta_{2.3} = 2A, \text{ etc.}$$

Also
$$\frac{E_1}{p \sin \theta_{2.3}} = \frac{E_2}{q \sin \theta_{3.1}} = \frac{E_3}{r \sin \theta_{1.2}}.$$

Hence
$$p \sin 2A = q \sin 2B = r \sin 2C.$$

It will be seen that three phase problems, considered graphically, generally resolve themselves into problems connected with the trigonometry of a triangle. When the arms of the load are inductive, then, as a rule, only approximate solutions can be got graphically, as the vectors can no longer be accurately represented geometrically (see Chapter VIII).

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The Electrician, Vol. 47, p. 639, 'The Elements of Three Phase Theory.' 1901.

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For Laplace's Method of Solving Functional Equations, see GEORGE BOOLE, 'The Calculus of Finite Differences.'

CHAPTER XII.

Two phase systems. The magnitudes and the phase differences of the P.D.'s between the mains. The voltage tetrahedron. The graphical representation of the voltages in the four arms of a star load. Rule for finding the voltages across the arms of a star load when the resistances of the arms are given. The voltages to the centre of the load so adjust themselves that the power expended on it is a minimum. The potentials to earth of the mains. The currents in a mesh load. The conditions under which it is possible for the P.D. waves between the four mains and earth to be similar curves. The P.D. waves between the mains are only similar in a special case. To find the phase differences between the opposite voltages and between the diagonal voltages in a two phase system. The measurement of power in two phase circuits. Two phase meters. When the currents in a star load are equal the sum of the voltages to the centre is a minimum.

Two phase systems.

DIAGRAMS 70 and 71 in the last chapter illustrate the fundamental principle of three phase machines with mesh and star wound armatures respectively. The principle of two phase machines is the same, but instead of the armature being wound in three sections it is wound in four. There is, however, in this case a third method of winding, which is illustrated in Fig. 88. *A*, *B*, *C* and *D* are the four terminals of the machine. *A* and *C* are the terminals of one winding of the armature and *B* and *D* are the terminals of the other. From symmetry the P.D.'s generated between *A* and *C* and between *B* and *D* when the field magnet revolves will differ in phase by ninety degrees. The currents flowing in circuits connected across the mains to *A* and *C* and in similar circuits connected across the mains to *B* and *D* will also differ in phase by a right angle and we shall see later on that it is easy to produce a rotary magnetic field by means of these currents, and hence two phase motors are simple to construct.

In practice instead of having four mains connected to A, B, C and D respectively it is customary to have only three, one of which is connected to two adjacent terminals, as for example A

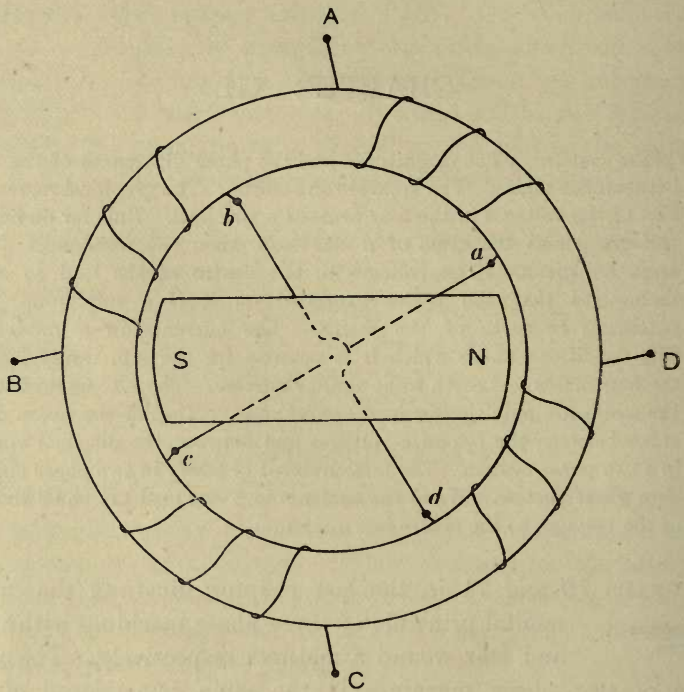

Fig. 88. Two Phase Machine with two separate windings on armature.

and B. In this case the P.D.'s between A and C and between B or A and D will still differ in phase by a right angle, and they will from symmetry be equal to one another. Now with our usual notation

$$\begin{aligned} v_{CD} &= v_C - v_D \\ &= -(v_A - v_C) - (v_D - v_A) \\ &= -v_{AC} - v_{DA}. \end{aligned}$$

Thus $$v_{CD} + v_{DA} + v_{AC} = 0,$$

and $$v_{CD} + v_{DB} + v_{CA} = 0.$$

Therefore $$V^2_{CD} = V^2_{DB} + V^2_{AC},$$

since V_{DB} and V_{AC} are in quadrature. They are also equal, and thus

$$V_{CD} = \sqrt{2}V_{DB}$$
$$= \sqrt{2}V_{AC}.$$

The main connected to A and B is generally referred to as the 'common return' and the other mains are sometimes referred to as the 'outers.' In Fig. 89, OX and OY represent the P.D.'s between the outers and the common return and OZ represents the P.D. between the two outers in magnitude and phase.

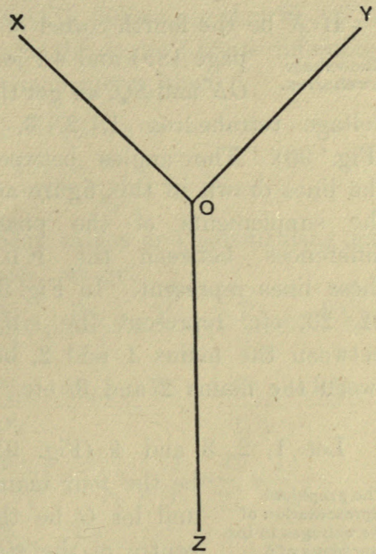

Fig. 89. Potential Differences in two phase three wire system with balanced load.

The magnitudes and the phase differences of the P.D.'s between the mains.

In the general case, that is, when four mains are used, it is impossible to represent the P.D.'s by vectors drawn in one plane. They can, however, be represented by vectors drawn in space.

Let v_1, v_2, v_3 and v_4 be the potentials of the four mains. Let $v_{1.2}$, $v_{2.3}$, $v_{3.4}$ and $v_{4.1}$ be the P.D.'s between the mains 1 and 2, 2 and 3, 3 and 4, and 4 and 1, respectively.

Then
$$v_{1.2} = v_1 - v_2,$$
$$v_{2.3} = v_2 - v_3,$$
$$v_{3.4} = v_3 - v_4,$$
$$v_{4.1} = v_4 - v_1.$$

Hence
$$v_{1.2} + v_{2.3} + v_{3.4} + v_{4.1} = 0.$$

A linear relation therefore connects the four instantaneous values, and hence by Chapter VIII their effective values can be represented by lines drawn in space. In Fig. 54, page 185, if we produce SO backwards to T making OT equal to OS, then OT will represent V_1 in this case, and the relations between the voltages and phase differences can be written down by the formulae given in that chapter.

The voltage tetrahedron.

If N be the fourth corner of the parallelogram ORQ (Fig. 54, page 185) and we join ON and SQ, we get the voltage tetrahedron 1, 2, 3, 4 (Fig. 90). The angles between the lines drawn in this figure are the supplements of the phase differences between the P. D.'s these lines represent. In Fig. 90 12, 23, etc. represent the P.D.'s between the mains 1 and 2, between the mains 2 and 3, etc.

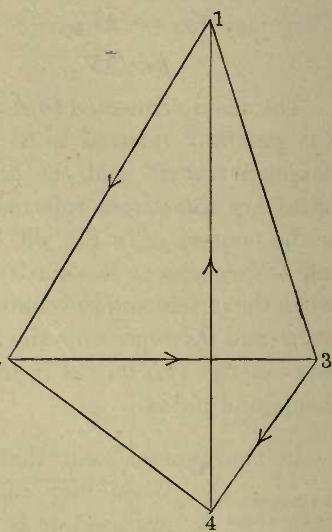

Fig. 90. The Voltage Tetrahedron.

The graphical representation of the voltages in the four arms of a star load.

Let 1, 2, 3 and 4 (Fig. 91) be the four mains and let O be the centre of the star load. Let r_1, r_2, r_3 and r_4 be the resistances (non-inductive) of the four arms, and let e_1, e_2, e_3 and e_4 be the P.D.'s between the mains and the centre of the load. Then, since the algebraical sum of the currents at O must be zero, we have

$$\frac{e_1}{r_1}+\frac{e_2}{r_2}+\frac{e_3}{r_3}+\frac{e_4}{r_4}=0 \ldots (1).$$

Hence $\frac{E_1}{r_1}$, $\frac{E_2}{r_2}$, $\frac{E_3}{r_3}$ and $\frac{E_4}{r_4}$ can be represented by vectors in the ways shown in Fig. 90 and in Fig. 54, page 185.

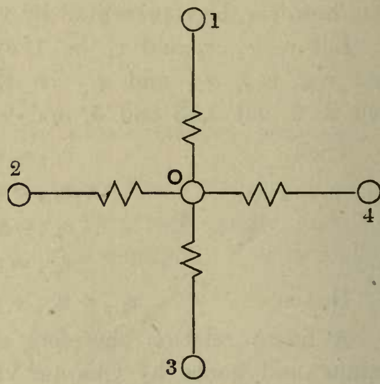

Fig. 91. Star Load.

We may write (1) in the form

$$e_1\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)=\frac{v_{1.2}}{r_2}+\frac{v_{1.3}}{r_3}+\frac{v_{1.4}}{r_4} \quad \ldots\ldots\ldots (2),$$

where $v_{1.2}$ = the P.D. between the mains 1 and 2

$$= e_1 - e_2.$$

Now by the voltage tetrahedron (Fig. 90) we see that $V_{1.2}$, $V_{2.3}$ and $V_{3.1}$ form a triangle. Hence if α be the angle of phase difference between $V_{1.2}$ and $V_{1.3}$ we find by trigonometry, that

$$V^2_{2.3} = V^2_{1.2} + V^2_{1.3} - 2V_{1.2}V_{1.3}\cos\alpha,$$

and thus
$$2\frac{V_{1.2}V_{1.3}\cos\alpha}{r_2r_3} = \frac{V^2_{1.2} + V^2_{1.3} - V^2_{2.3}}{r_2r_3}.$$

Hence, squaring (2), taking mean values and substituting for the cosines of the phase differences, we get

$$E_1^2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 = \frac{V^2_{1.2}}{r_2^2} + \frac{V^2_{1.3}}{r_3^2} + \frac{V^2_{1.4}}{r_4^2} + \frac{V^2_{1.2}+V^2_{1.3}-V^2_{2.3}}{r_2r_3}$$
$$+ \frac{V^2_{1.3}+V^2_{1.4}-V^2_{3.4}}{r_3r_4} + \frac{V^2_{1.4}+V^2_{1.2}-V^2_{2.4}}{r_4r_2}$$
$$= \left\{\frac{V^2_{1.2}}{r_2} + \frac{V^2_{1.3}}{r_3} + \frac{V^2_{1.4}}{r_4}\right\}\left\{\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right\}$$
$$- \frac{V^2_{2.3}}{r_2r_3} - \frac{V^2_{3.4}}{r_3r_4} - \frac{V^2_{4.2}}{r_4r_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The other three equations giving E_2, E_3 and E_4 in terms of the six P.D.'s between the mains can easily be written down by symmetry. By adding up these four equations and cancelling out the common factor we deduce

$$\left(\frac{E_1^2}{r_1}+\frac{E_2^2}{r_2}+\frac{E_3^2}{r_3}+\frac{E_4^2}{r_4}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)$$
$$= \frac{V^2_{1.2}}{r_1r_2} + \frac{V^2_{1.3}}{r_1r_3} + \frac{V^2_{1.4}}{r_1r_4} + \frac{V^2_{2.3}}{r_2r_3} + \frac{V^2_{2.4}}{r_2r_4} + \frac{V^2_{3.4}}{r_3r_4} \quad \ldots\ldots(4).$$

If the resistances are all equal

$$4(E_1^2 + E_2^2 + E_3^2 + E_4^2)$$
$$= V^2_{1.2} + V^2_{1.3} + V^2_{1.4} + V^2_{2.3} + V^2_{2.4} + V^2_{3.4} \quad \ldots\ldots\ldots(5).$$

If, in addition, the voltages between adjacent mains are all equal and the diagonal voltages, namely $V_{1.3}$ and $V_{2.4}$, are also equal, then from (3)

$$E_1 = E_2 = E_3 = E_4.$$

And from (5) $$8E_1^2 = 2V^2_{1.2} + V^2_{1.3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

If, finally, $V_{1.2}$ and $V_{2.3}$ are in quadrature

$$V^2_{1.3} = V^2_{1.2} + V^2_{2.3}$$
$$= 2V^2_{1.2}.$$

Hence $$\sqrt{2}E_1 = V_{1.2} \quad \text{..............................(7).}$$

In getting the equation (7) we have made no assumption as to the shapes of the waves of P.D. but the shapes are restricted by equation (1).

Looking back at Fig. 54, page 185, we see that if OP, OQ, OR and OT (drawn equal and opposite to OS) represent forces acting at O, they will be in equilibrium. Now, if G be the centre of gravity of equal particles placed at P, Q, R and T, the resultant of the forces OP, OQ, OR and OT will be $4 . OG$; but since they are in equilibrium, this resultant must be zero, and therefore G must coincide with O. Hence O is the centre of gravity of equal masses placed at P, Q, R and T.

Let OP represent $\frac{E_1}{r_1}$, and let OQ, OR and OT represent $\frac{E_2}{r_2}$, $\frac{E_3}{r_3}$ and $\frac{E_4}{r_4}$. Then, since the resistances are non-inductive, $r_1 . OP$, $r_2 . OQ$, $r_3 . OR$ and $r_4 . OT$ represent E_1, E_2, E_3 and E_4 respectively. Let $OP' = r_1 . OP$, $OQ' = r_2 . OQ$, etc., then $P'Q'R'T'$ will be the tetrahedron that gives the P.D.'s between the mains. To prove this it is sufficient to notice that

$$v_{1.2} = e_1 - e_2.$$

Therefore $$V^2_{1.2} = E_1^2 + E_2^2 - 2E_1E_2 \cos \phi_{1.2},$$

and hence the rest follows by trigonometry.

Rule for finding the voltages across the arms of a star load when the resistances of the arms are given.

Construct the tetrahedron that gives the voltages between the mains. In order to do this we need to take six voltage readings, namely, the four readings between adjacent mains and the two diagonal readings. Find the centre of gravity G of masses $\frac{1}{r_1}$, $\frac{1}{r_2}$, $\frac{1}{r_3}$ and $\frac{1}{r_4}$ placed at the four angular points of the tetrahedron. Then the lines joining G to the four angular points will give the phase differences and the magnitudes of the required voltages.

The voltages to the centre of the load so adjust themselves that the power expended on it is a minimum.

Let L, M, N and R be the angular points of the tetrahedron representing the voltages between the mains, and we suppose that the dynamo maintains these voltages constant. Then, by a well-known theorem in statics, if O be any point in space

$$\frac{OL^2}{r_1}+\frac{OM^2}{r_2}+\frac{ON^2}{r_3}+\frac{OR^2}{r_4}$$

is a minimum when O is the centre of gravity of masses $\frac{1}{r_1}$, $\frac{1}{r_2}$, $\frac{1}{r_3}$ and $\frac{1}{r_4}$ placed at L, M, N and R respectively. But since the resistances are non-inductive, this expression represents the power expended on the load, and hence the theorem follows.

The potentials to earth of the mains.

Let f_1, f_2, f_3 and f_4 be the fault resistances of the four mains; then, by Kirchhoff's law when the condenser currents in the sheath are negligible, we have

$$\frac{v_1}{f_1}+\frac{v_2}{f_2}+\frac{v_3}{f_3}+\frac{v_4}{f_4}=0.$$

Proceeding as above, we see that, if G be the centre of gravity of masses $\frac{1}{f_1}$, $\frac{1}{f_2}$, $\frac{1}{f_3}$ and $\frac{1}{f_4}$ placed at the vertices of the voltage tetrahedron $LMNR$, then GL, GM, GN and GR represent V_1, V_2, V_3 and V_4 respectively in magnitude and phase.

It follows that if the fault resistances of the mains vary, then their potentials adjust themselves so that the power lost in leakage currents is a minimum.

The currents in a mesh load.

Let a_1, a_2, a_3 and a_4 (Fig. 92) be the currents in the mains, and let i_1, i_2, i_3 and i_4 be the currents in the mesh load.

Then
$$\left.\begin{aligned} a_1 &= i_1 - i_4, & a_2 &= i_2 - i_1 \\ a_3 &= i_3 - i_2, & a_4 &= i_4 - i_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Therefore
$$a_1+a_2+a_3+a_4=0.$$

This also follows at once since there is no accumulation of current in the load.

Hence, like the voltages, A_1, A_2, A_3 and A_4 can be represented by lines drawn from a point in space, and this point is the centre

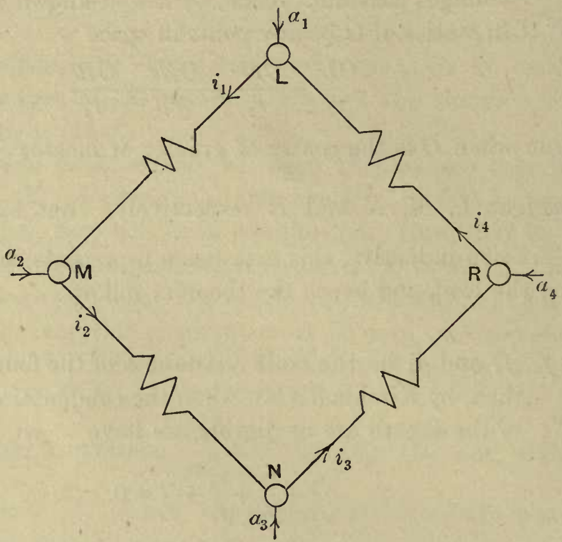

Fig. 92. The Currents in a Mesh Load.

of gravity of four equal particles placed at their extremities. The currents can also be represented by a skew quadrilateral, the angles of which are the supplements of the angles of phase difference between the four currents.

Again, if r_1, r_2, r_3 and r_4 be the resistances of the arms *LM*, *MN*, *NR* and *RL* in Fig. 92, then

$$\left.\begin{array}{ll} r_1\dot{i}_1 = v_1 - v_2, & r_2\dot{i}_2 = v_2 - v_3 \\ r_3\dot{i}_3 = v_3 - v_4, & r_4\dot{i}_4 = v_4 - v_1 \end{array}\right\}.$$

Therefore $$r_1\dot{i}_1 + r_2\dot{i}_2 + r_3\dot{i}_3 + r_4\dot{i}_4 = 0 \quad\quad (2).$$

Hence, if r_1I_1, r_2I_2, r_3I_3 and r_4I_4 be represented by lines drawn from a point O, this point will be the centre of gravity of equal masses placed at the extremities of these lines. The lines joining these extremities give the voltage tetrahedron of the P.D.'s between the mains.

Again, if we draw I_1, I_2, I_3 and I_4 (Fig. 93) from the point O, it can easily be proved that O is the centre of gravity of particles, whose masses are proportional to r_1, r_2, r_3 and r_4, placed at the extremities of I_1, I_2, I_3 and I_4. Equations (1) show us that the lines joining the extremities of I_1 and I_4, I_2 and I_1, I_3 and I_2 and I_4 and I_3 represent A_1, A_2, A_3 and A_4.

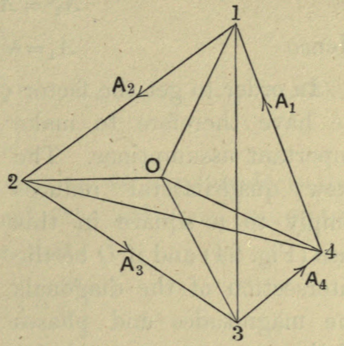

Fig. 93. The Current Tetrahedron.

Let (Fig. 93) A_5 be equal to the length of the line joining 1 to 3 and let A_6 be the length of the line joining 2 to 4, then

$$A_5^2 = A_2^2 + A_3^2 + 2A_2A_3 \cos \phi_{2.3},$$

and

$$A_6^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \phi_{1.2}.$$

An inspection of Fig. 93 will show that there are several other expressions for A_5 and A_6.

From (2)

$$(r_1 + r_2 + r_3 + r_4)\, i_1 = r_2 i_{1.2} + r_3 i_{1.3} + r_4 i_{1.4},$$

where

$$i_{1.2} = i_1 - i_2 = -a_2, \text{ etc.}$$

Proceeding in exactly the same way as we did with the corresponding voltage equation, we find

$$I_1^2 (r_1 + r_2 + r_3 + r_4)^2 = (r_2A_2^2 + r_3A_5^2 + r_4A_1^2)(r_2 + r_3 + r_4) - r_2r_3A_3^2 - r_3r_4A_4^2 - r_4r_2A_6^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and three similar equations.

Hence also

$$(I_1^2 r_1 + I_2^2 r_2 + I_3^2 r_3 + I_4^2 r_4)(r_1 + r_2 + r_3 + r_4) = r_1r_2A_2^2 + r_1r_3A_5^2 + r_1r_4A_1^2 + r_2r_3A_3^2 + r_2r_4A_6^2 + r_3r_4A_4^2 \ldots (4).$$

If the resistances are all equal, then

$$4(I_1^2 + I_2^2 + I_3^2 + I_4^2) = A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + A_6^2 \ldots (5).$$

If, in addition, the currents in the mains are all equal and the phases are such that A_5 equals A_6, then from (3)

$$I_1 = I_2 = I_3 = I_4.$$

Hence

$$8I_1^2 = 2A_1^2 + A_5^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

If the currents in the mains are in quadrature, then (Fig. 93)

$$A_5^2 = A_2^2 + A_3^2 = 2A_1^2.$$

Hence $$A_1 = \sqrt{2} I_1 \quad \text{.................................(7)}.$$

In order to get the factor $\sqrt{2}$, which is used in practical work, we have therefore to make important assumptions. The skew quadrilateral reduces simply to a square in this case (Fig. 94) and if O be the intersection of the diagonals the magnitudes and phases of the currents are as shown in this figure.

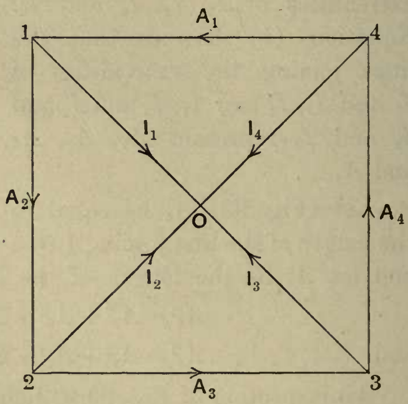

Fig. 94. The currents in a balanced load.

The conditions under which it is possible for the P. D. waves between the four mains and earth to be similar curves.

Let the equation to the wave of P.D. between a main and earth be of the form

$$e = Vf(t).$$

Let r_1, r_2, r_3 and r_4 be the fault resistances of the mains and let their potentials be

$$V_1 f(t), \quad V_2 f\left(t + \frac{T}{4}\right), \quad V_3 f\left(t + \frac{T}{2}\right) \text{ and } V_4 f\left(t + \frac{3T}{4}\right)$$

respectively, then

$$\frac{V_1}{r_1} f(t) + \frac{V_2}{r_2} f\left(t + \frac{T}{4}\right) + \frac{V_3}{r_3} f\left(t + \frac{T}{2}\right) + \frac{V_4}{r_4} f\left(t + \frac{3T}{4}\right) = 0 \ldots (1).$$

Now this is true for all values of t, hence writing $t + \frac{T}{4}$, $t + \frac{T}{2}$, and $t + \frac{3T}{4}$ for t in (1) successively, we get four equations from which the functions can easily be eliminated by determinants and we get that

$$\left(\frac{V_1}{r_1} + \frac{V_2}{r_2} + \frac{V_3}{r_3} + \frac{V_4}{r_4}\right)\left(\frac{V_1}{r_1} - \frac{V_2}{r_2} + \frac{V_3}{r_3} - \frac{V_4}{r_4}\right)$$
$$\left\{\left(\frac{V_1}{r_1} - \frac{V_3}{r_3}\right)^2 + \left(\frac{V_2}{r_2} - \frac{V_4}{r_4}\right)^2\right\} = 0.$$

Hence either $$\frac{V_1}{r_1}+\frac{V_3}{r_3}=\frac{V_2}{r_2}+\frac{V_4}{r_4} \qquad (2),$$

or $$\left.\begin{aligned}\frac{V_1}{r_1}&=\frac{V_3}{r_3}\\ \frac{V_2}{r_2}&=\frac{V_4}{r_4}\end{aligned}\right\} \qquad (3).$$

and

In what precedes the only assumption we have made is that $f(t)$ is a periodic function. If we make the additional assumption that it is an alternating function, *i.e.* that

$$f(t)=-f\left(t+\frac{T}{2}\right)$$

and $$f\left(t+\frac{T}{4}\right)=-f\left(t+\frac{3T}{4}\right),$$

we get from (1) that

$$\left(\frac{V_1}{r_1}-\frac{V_3}{r_3}\right)f(t)+\left(\frac{V_2}{r_2}-\frac{V_4}{r_4}\right)f\left(t+\frac{T}{4}\right)=0.$$

As this has to be true for all values of t, we must have

$$\frac{V_1}{r_1}=\frac{V_3}{r_3} \text{ and } \frac{V_2}{r_2}=\frac{V_4}{r_4} \qquad (3).$$

Now V_1, V_2, V_3 and V_4 are determined by finding the centre of gravity of masses $\frac{1}{r_1}$, $\frac{1}{r_2}$, $\frac{1}{r_3}$ and $\frac{1}{r_4}$ placed at the angular points of the voltage tetrahedron and joining this point to the angular points, hence (3) can only be true in very special cases. We are therefore not justified in assuming that the potential waves between the mains and earth are similar curves.

Suppose that the P.D. waves between the mains are

The P.D. waves between the mains are only similar in a special case.

$$V_1 f(t),\ V_2 f\left(t+\frac{T}{4}\right),\ V_3 f\left(t+\frac{T}{2}\right) \text{ and } V_4 f\left(t+\frac{3T}{4}\right).$$

Then since the sum of them must be zero and

$$f(t)=-f\left(t+\frac{T}{2}\right)$$

we get $$(V_1-V_3)f(t)+(V_2-V_4)f\left(t+\frac{T}{4}\right)=0.$$

Hence V_1 must equal V_3 and V_2 must equal V_4.

To find the phase differences between the opposite voltages and between the diagonal voltages in a two phase system.

Let *LMNR* (Fig. 95) be the voltage tetrahedron. Let *A*, *B*, *C*, *D*, *E* and *F* be the middle points of its edges. Then from geometry we see that *ABCD*, etc. are parallelograms, and that *AC*, *BD* and *EF* intersect in a point which is the centre of gravity of the tetrahedron. We will first find ϕ the angle of phase difference between the diagonal voltages *LN* and *MR*.

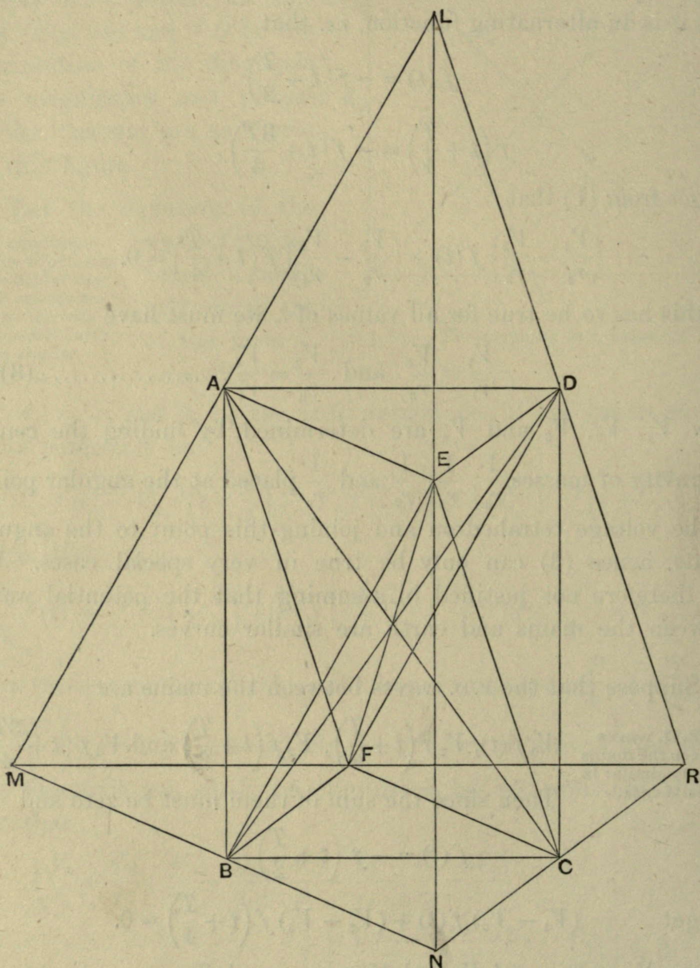

Fig. 95. Finding the Phase Differences in a Two Phase System.

We have

$$2LN \, . \, MR \cos\phi = 4 \, . \, 2AB \, . \, BC \cos ABC$$
$$= 2(AC^2 - BD^2)$$
$$= 2\{AC^2 + EF^2 - (BD^2 + EF^2)\}$$
$$= 4\{AE^2 + EC^2 - (EB^2 + ED^2)\}$$
$$= MN^2 + LR^2 - (LM^2 + NR^2),$$

and therefore $$\cos\phi = \frac{V^2_{2.3} + V^2_{4.1} - (V^2_{1.2} + V^2_{3.4})}{2V_{1.3}V_{2.4}} \quad \text{............(1)}.$$

Similarly if ϕ' be the angle of phase difference between the opposite voltages $V_{1.2}$ and $V_{3.4}$, then

$$\cos\phi' = \frac{V^2_{1.3} + V^2_{2.4} - (V^2_{2.3} + V^2_{4.1})}{2V_{1.2}V_{3.4}} \quad \text{............(2)}.$$

The phase differences between the other voltages can easily be got from the triangular faces of the voltage tetrahedron (Fig. 95).

If the diagonal voltages be in quadrature

$$V^2_{2.3} + V^2_{4.1} = V^2_{1.2} + V^2_{3.4} \quad \text{..................(3)}.$$

Conversely, if (3) be true, then from (1), the phase difference of the diagonal voltages is 90 degrees.

If $V_{1.2}$ and $V_{3.4}$ (LM and NR in the figure) be in opposition in phase, then from (2)

$$V^2_{1.3} + V^2_{2.4} = V^2_{2.3} + V^2_{4.1} + 2V_{1.2}V_{3.4} \quad \text{............(4)}.$$

In this case, since LM and NR are parallel, the four points L, M, N and R are in one plane, and (4) could easily be proved otherwise.

If $V_{1.2}$ and $V_{3.4}$ and also $V_{2.3}$ and $V_{4.1}$ be in opposite phases, then $V_{1.2} = V_{3.4}$ and $V_{2.3} = V_{4.1}$. If, in addition, $V_{1.3}$ and $V_{2.4}$ be in quadrature, then

$$V_{1.2} = V_{2.3} = V_{3.4} = V_{4.1} = \frac{V_{1.3}}{\sqrt{2}} = \frac{V_{2.4}}{\sqrt{2}}.$$

The measurement of power in two phase circuits.

Let 1, 2, 3 and 4 (Fig. 96) be the mains and suppose that there is both a mesh winding and a star winding, the centre of the star winding being O.

Let a_1, a_2, a_3 and a_4 be the currents in the mains, $i_{1.2}$ the current in the mesh winding joining 1 and 2, and let $\dot{i}_1$ be the current in the star winding from 1 to O. Then

$$\left.\begin{aligned} a_1 &= \dot{i}_1 + \dot{i}_{1.2} - \dot{i}_{4.1} \\ a_2 &= \dot{i}_2 + \dot{i}_{2.3} - \dot{i}_{1.2} \\ a_3 &= \dot{i}_3 + \dot{i}_{3.4} - \dot{i}_{2.3} \\ a_4 &= \dot{i}_4 + \dot{i}_{4.1} - \dot{i}_{3.4} \end{aligned}\right\} .$$

Therefore $$a_1 + a_2 + a_3 + a_4 = 0.$$

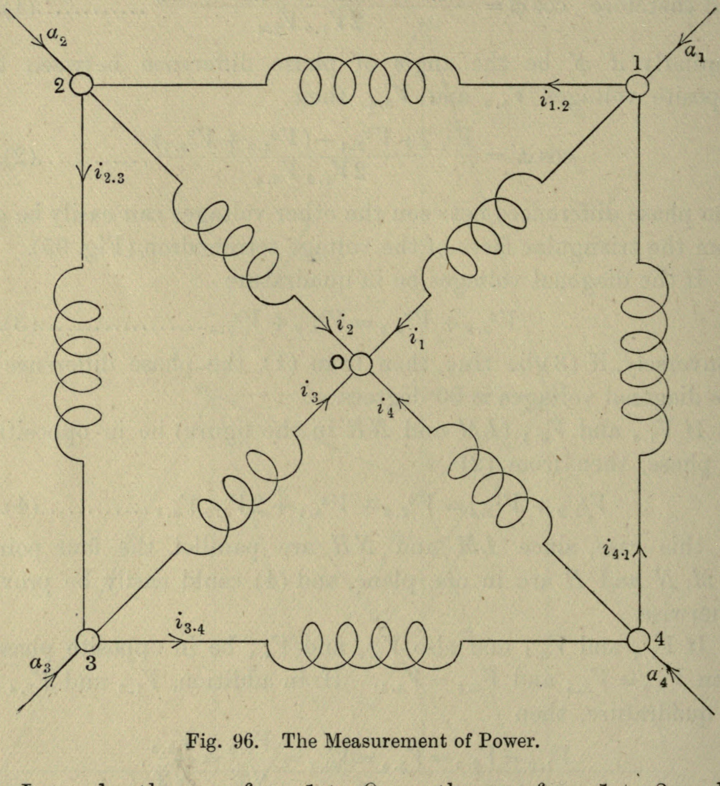

Fig. 96. The Measurement of Power.

Let v_1 be the P.D. from 1 to O, $v_{1.2}$ the P.D. from 1 to 2, and w the power expended in the windings. Then

$$\begin{aligned} w &= v_1\dot{i}_1 + v_2\dot{i}_2 + v_3\dot{i}_3 + v_4\dot{i}_4 \\ &\qquad + (v_1 - v_2)\,\dot{i}_{1.2} + (v_2 - v_3)\,\dot{i}_{2.3} + (v_3 - v_4)\,\dot{i}_{3.4} + (v_4 - v_1)\,\dot{i}_{4.1} \\ &= v_1a_1 + v_2a_2 + v_3a_3 + v_4a_4 \quad \ldots\ldots\ldots\ldots (1), \\ &= v_{1.4}a_1 + v_{2.4}a_2 + v_{3.4}a_3 \quad \ldots\ldots\ldots\ldots (2). \end{aligned}$$

Formulae (1) and (2) give the two methods of measuring power in two phase circuits. For the first method we require four wattmeters, the ampere coils being put in series with the mains, and the volt coils being connected from the mains to the centre of the star system. For the second method we require three wattmeters. The ampere coils are put in any three of the mains and the volt coils of the three wattmeters are connected from these mains with the other main. Care must be taken to find out whether any of the wattmeters are reading negatively.

If the system is symmetrical, then

$$w = 4v_1 a_1 \text{ .... } (3),$$

and

$$w = 2v_{1.4}a_1 + v_{2.4}a_2 \text{ .... } (4).$$

Hence one wattmeter is required for the first method, the multiplying factor being 4, and for the second two wattmeters are required. Their ampere coils are put in adjacent mains, No. 1 and No. 2 for example, and their volt coils are connected between No. 1 and No. 4 and No. 2 and No. 4 respectively. Twice the reading of the first wattmeter added to the reading of the second will give us the watts expended in the circuit. The first of these methods is obviously the preferable one.

When part of the load is in series with the mains as in Fig. 83 Chapter XI, the formulae still apply. The proof is practically the same as for the three phase case.

Two phase meters.

Just as in the case of three phase circuits, if we have an ordinary watt-hour meter and the arms are equally loaded, we can measure the energy consumed by connecting up as in Fig. 97, where S is a star-box connected with the four mains. The energy recorded multiplied by four will give the total energy expended in the given time. If the arms are unequally loaded, we should require a watt-hour meter in each arm and the sum of the meter readings would give the required energy.

If we connect up as in Fig. 98, then the sum of the three readings will measure the units consumed. Thus we can make a two phase meter with six ampere terminals and one volt terminal. If the arms are equally loaded and the load be non-inductive, then the reading of either of the meters AV or $A''V''$ (Fig. 98) multiplied by four or the reading of the meter $A'V'$ multiplied by two will give the required energy.

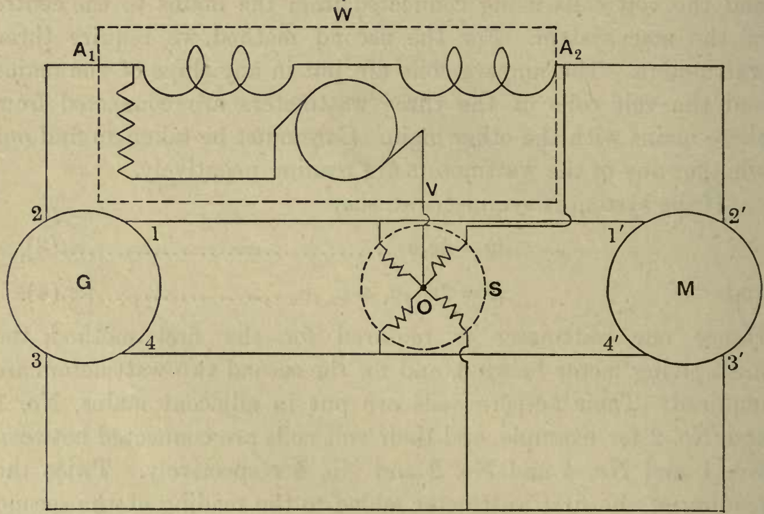

Fig. 97. Connections of watt-hour meter for balanced working. A_1 and A_2 are the ampere terminals of the meter and V is the volt terminal. G is the generator, M the motor and O the centre of the star-box S.

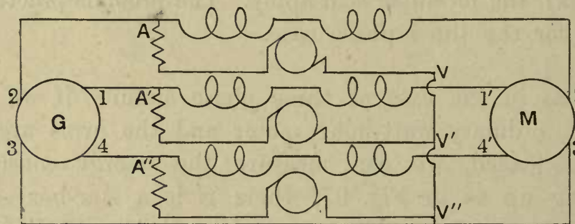

Fig. 98. Connections of three watt-hour meters for measuring the energy expended in any two phase circuit.

Fig. 99 shows the connections for the case of a common return. The sum of the two watt-hour meter readings will obviously give the units consumed. The two meters could be combined into a single instrument having four ampere terminals and one volt terminal which has to be connected to the common return. By comparison with Fig. 86, Chapter XI, it will be seen that this form of meter measures the units consumed either on three phase circuits or on two phase circuits with a common return. It would also measure the units consumed in a three wire direct current system.

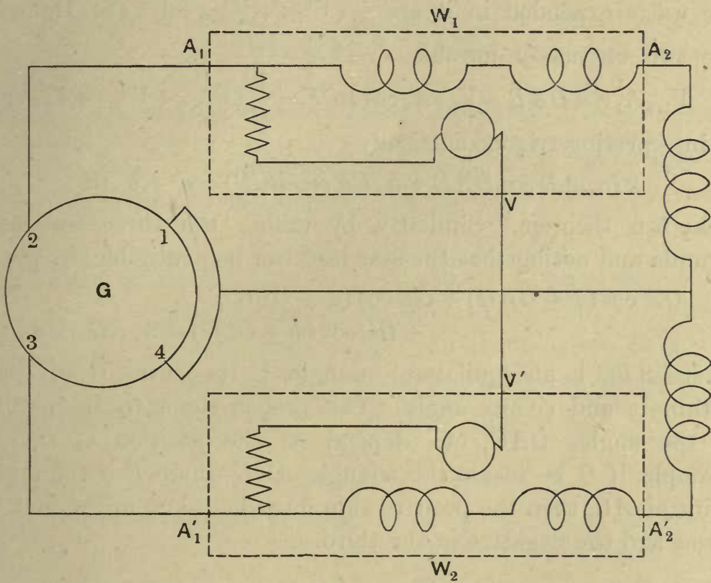

Fig. 99. Two meters sufficient when there is a common return.

When the currents in a star load are equal the sum of the voltages to the centre is a minimum.

Let $LMNR$ be the voltage tetrahedron, the magnitude of which we suppose fixed, and O any point, then $OL + OM + ON + OR$ is a minimum when any two of the opposite edges subtend equal angles at O. In this case the solid angles (see Chapter VIII) formed by any three of the vectors OL, OM, ON and OR are equal, and hence

$$\frac{OL}{r_1} = \frac{OM}{r_2} = \frac{ON}{r_3} = \frac{OR}{r_4},$$

i.e. the currents in the four arms of the star load are equal.

It is interesting to note that the equations for the electrical methods of measuring power in two and three phase circuits, when interpreted geometrically, give rise to numerous theorems connected with the trigonometry of triangles and tetrahedrons.

For example, if ABC be the voltage triangle for a three phase system and D, E and F are the middle points of the sides, then if R be the resistance (non-inductive) of each arm of a mesh load, the watts expended in it are $\frac{1}{R}(V^2_{1.2}+V^2_{2.3}+V^2_{3.1})$. Hence by the two wattmeter formula

$$V_{1.2}A_1 \cos DAB + V_{3.2}A_3 \cos BCF = \frac{1}{R}(V^2_{1.2}+V^2_{2.3}+V^2_{3.1})$$

or interpreting trigonometrically

$$2(c\,.\,AD \cos DAB + a\,.\,CF \cos BCF) = a^2+b^2+c^2,$$

a known theorem. Similarly, by taking the three wattmeter formula and noting that the star load can be negligible, we get

$$OA\cos(\phi \pm OAG) + OB\cos(\phi \pm OBG) + OC\cos(\phi \pm OCG) = 3\,.\,AG\cos\phi,$$

where ABC is an equilateral triangle, G its centre, O any point within it and ϕ any angle. The proper signs to be prefixed to the angles OAG, etc. depend on the position of O. For example, if O be inside the triangle AGE where E is the middle point of AC, then the positive sign must be taken in the first two terms and the negative in the third.

CHAPTER XIII.

Conversion of a two phase system to a three phase system. Conversion of a three phase system to a two phase system. The power factor of a three phase system. Wattmeter method of finding cos ϕ. Phase indicator. Meter for the wattless current. Induction type watt-hour meters. References.

Conversion of a two phase system to a three phase system.

THE following method, due to C. F. Scott, of converting a two phase system of supply into a three phase system is sometimes useful. In the diagram (Fig. 100) 1 and 3, 2 and 4 are the terminals of the primaries of two transformers which are connected to the two phase mains.

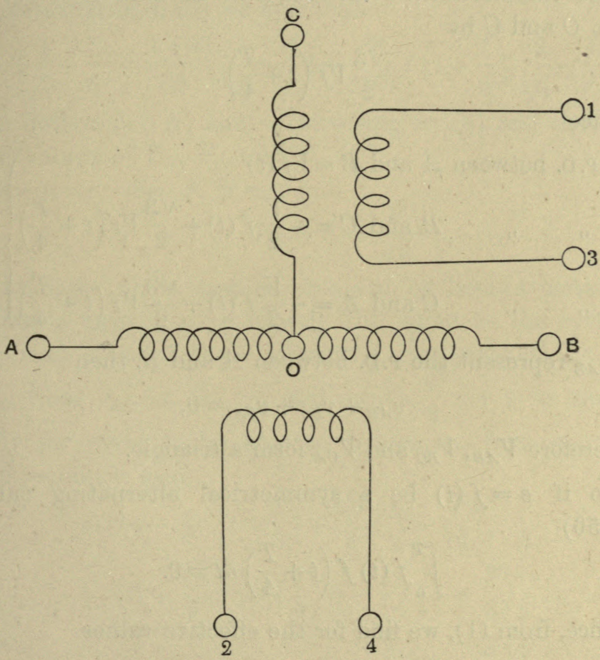

Fig. 100. Two phase to three phase transformer. Scott's method.

AOB and OC are the secondaries of the two transformers, O being the middle point of AB. If the number of windings be so arranged that the P.D. between A and B is to the P.D. between C and O in the ratio of 2 to $\sqrt{3}$, then the P.D.'s between A and B, B and C, and C and A will be equal and the phase differences between these P.D.'s will also be equal. The following investigation shows how the new P.D. wave shape may differ from the old P.D. wave shape and also gives a rigorous proof of the method.

We will suppose that there is no magnetic leakage and that the resistances of the primary and secondary windings are negligible. In this case the ratio of the potential difference applied to the primary to the potential difference between the secondary terminals will be equal to the ratio of the number of turns in the primary to the number of turns in the secondary. We will also suppose that the P.D. between 1 and 3 is a quarter of a period in advance of the period of the P.D. between 2 and 4. We can represent therefore the P.D. between A and B by $Vf(t)$ and between O and C by

$$\frac{\sqrt{3}}{2} Vf\left(t + \frac{T}{4}\right).$$

Hence

$$\left.\begin{array}{ll} \text{the P.D. between } A \text{ and } B = Vf(t) & \\ \quad ,, \quad ,, \quad B \text{ and } C = -\frac{V}{2} f(t) + \frac{\sqrt{3}}{2} Vf\left(t + \frac{T}{4}\right) & \\ \quad ,, \quad ,, \quad C \text{ and } A = -\frac{V}{2} f(t) - \frac{\sqrt{3}}{2} Vf\left(t + \frac{T}{4}\right) & \end{array}\right\} \ldots (1).$$

If v_{AB} represent the P.D. between A and B, then

$$v_{AB} + v_{BC} + v_{CA} = 0,$$

and therefore V_{AB}, V_{BC} and V_{CA} form a triangle.

Also if $e = f(t)$ be a symmetrical alternating curve (see page 156)

$$\int_0^T f(t) f\left(t + \frac{T}{4}\right) dt = 0.$$

Hence, from (1), we find for the effective values

$$V_{AB} = V_{BC} = V_{CA} = V \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The voltage triangle is therefore equilateral and the phase differences are 120 degrees.

It is impossible to determine the values of v_A, v_B and v_C from (1), but we can write

$$\left.\begin{aligned} v_A &= \frac{V}{2} f(t) + \frac{V}{2\sqrt{3}} f\left(t+\frac{T}{4}\right) + K \\ v_B &= -\frac{V}{2} f(t) + \frac{V}{2\sqrt{3}} f\left(t+\frac{T}{4}\right) + K \\ v_C &= -\frac{V}{\sqrt{3}} f\left(t+\frac{T}{4}\right) + K \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3),$$

where K is a constant or a function of the time.

Let the fault resistances of the three mains connected with A, B and C be f_1, f_2 and f_3 respectively.

Then, when we can neglect the capacity current in the sheath,

$$\frac{v_A}{f_1} + \frac{v_B}{f_2} + \frac{v_C}{f_3} = 0.$$

Substituting from (3) we find that

$$K\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) = \frac{Vf(t)}{2}\left(\frac{1}{f_2} - \frac{1}{f_1}\right) + \frac{V}{2\sqrt{3}} f\left(t+\frac{T}{4}\right)\left\{\frac{2}{f_3} - \frac{1}{f_1} - \frac{1}{f_2}\right\}.$$

This determines K, and substituting in (3) and squaring we find the values of V_A, V_B and V_C.

If $f_1 = f_2 = f_3$, then K is zero and

$$V_A = V_B = V_C = \frac{V}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If the three phase system, obtained by Scott's method from a two phase supply, were exactly the same as the system got directly from a three phase alternator, we ought to be able to get v_{AB} from v_{BC} in (1) by writing $t - \frac{T}{3}$ for t in the latter. Similarly we ought to be able to deduce v_{BC} from v_{CA}.

Thus, we must have

$$\left.\begin{aligned} f(t) &= -\tfrac{1}{2} f\left(t-\frac{T}{3}\right) + \frac{\sqrt{3}}{2} f\left(t-\frac{T}{12}\right), \\ \text{and} \quad -\tfrac{1}{2} f(t) + \frac{\sqrt{3}}{2} f\left(t+\frac{T}{4}\right) &= -\tfrac{1}{2} f\left(t-\frac{T}{3}\right) \\ &\quad - \frac{\sqrt{3}}{2} f\left(t-\frac{T}{12}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots(5).$$

These equations are only true for special values of $f(t)$.

We saw, on page 233, that, with a symmetrically made three phase machine, only the $(6n \pm 1)$th harmonics can be present. In an ordinary two phase machine all the odd harmonics may be present, and hence $f(t)$ may contain the third and other harmonics which are not present in the waves from a three phase machine. If, therefore, the Scott system is to yield currents similar to those obtainable from a three phase machine, a first essential condition is that the two phase alternator must be such that the third, ninth, fifteenth ... harmonics are absent from its electromotive force wave. Equations (5) give the other essential condition.

Again, when the system is balanced, K is zero and we ought to be able to get v_A from v_B in (3) by writing $t - \frac{T}{3}$ for t in the latter. Similarly we ought to be able to deduce v_B from v_C.

Hence

$$-\frac{1}{2}f\left(t-\frac{T}{3}\right)+\frac{1}{2\sqrt{3}}f\left(t-\frac{T}{12}\right)=\frac{1}{2}f(t)+\frac{1}{2\sqrt{3}}f\left(t+\frac{T}{4}\right),$$

or

$$\left.\begin{aligned} \sqrt{3}\left\{f(t)+f\left(t-\frac{T}{3}\right)\right\} &= f\left(t-\frac{T}{12}\right)-f\left(t+\frac{T}{4}\right). \\ \text{Similarly}\quad 2f\left(t-\frac{T}{12}\right) &= \sqrt{3}\,f(t)-f\left(t+\frac{T}{4}\right). \end{aligned}\right\}\ldots(6).$$

We can verify that

$$f(t) = X' \sin\left(\frac{2\pi t}{T} + Y'\right)$$

is a solution of the equations (5) and (6), where X' and Y' are constants or functions of t that do not alter when $t + \frac{T}{12}$ is substituted for t in them. Comparing this solution with the solution arrived at for a symmetrical three phase system (page 231) we see that the only difference is that it is possible for X and Y to have double the period of X' and Y'.

As a general rule, therefore, the shape of the waves in a three phase system obtained by Scott's method will not be similar to the shape of those obtained from a three phase machine. In practice, however, this is a matter of little importance. The only objection to the method is that there can be a greater number of harmonics in the electromotive force waves, and thus the chance of resonance occurring at particular frequencies may be greater. The dangers from resonance arise from the high potential differences set up in various parts of the distributing net-work. The insulation of cables, transformers and armature windings, etc. is sometimes broken down from this cause.

Conversion of a three phase system to a two phase system.

If we connect A, B and C (Fig. 100) with the three phase mains, we can obviously get two phase currents from 1, 3 and 2, 4. There are other ways of doing this which are suggested by elementary geometrical considerations.

Scott's arrangement can also be generalised without difficulty. For example, if the primaries of four transformers be magnetised by the currents in the arms of a two phase mesh load and each transformer have two secondary windings, the middle points of one set being attached to four terminals, then we could either have four sets of three phase terminals, two sets of six phase terminals, or two sets of three phase terminals and one set of two phase terminals.

Let the terminals of the two secondary windings of the transformer, the primary coil of which forms the first arm of the two phase mesh load, be A_1, A_1' and B_1, B_1' respectively, and let the ratio of the turns of wire in the coil A_1A_1' to the turns in the coil B_1B_1' be as 2 is to $\sqrt{3}$. Let also a terminal O_1 on this transformer be connected with the middle of the coil A_1A_1'. If the primary coils of three transformers similar to this, form the other arms of the two phase load, then we shall have twenty secondary terminals A_1, A_2, A_3, A_4, B_1, ... If we join O_1 and B_2, then by the theorem given above A_1, A_1' and B_2' are three phase terminals; and if we join O_3 and B_4, then A_3, A_3' and B_4' are also three phase terminals. It is easy to see that a six phase star load or a six phase mesh load can be connected across A_1, B_4', A_1', A_3, B_2' and A_3'. Similarly many other combinations may be obtained from the twenty terminals.

The power factor of a three phase system.

Consider first the case of a mesh connected load. Let W_1, W_2 and W_3 be the power expended in the three arms, and let I_1, I_2 and I_3 be the effective currents in them. Then, if $\cos\phi_1$, $\cos\phi_2$ and $\cos\phi_3$ be the power factors of the three arms, we have

$$\cos\phi_1 = \frac{W_1}{V_{2.3}I_1}, \quad \cos\phi_2 = \frac{W_2}{V_{3.1}I_2}, \quad \cos\phi_3 = \frac{W_3}{V_{1.2}I_3}.$$

The power factor ($\cos\phi$) of the load may be defined as the ratio of the true watts to the sum of the apparent watts, and hence

$$\cos\phi = \frac{W_1 + W_2 + W_3}{V_{2.3}I_1 + V_{3.1}I_2 + V_{1.2}I_3} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

It therefore lies in value between the greatest and the least values of the power factors of the arms.

When the load is balanced,

$$\cos\phi = \frac{W_1 + W_2 + W_3}{\sqrt{3}V_{2.3}A_1} = \frac{W}{\sqrt{3}VA} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

where W represents the power expended in the balanced load, V the mesh voltage and A the current in a main.

In order to find $\cos\phi$ we must therefore measure the power expended in the load by the two wattmeter method; if a star-box is available one wattmeter will be sufficient. We must also measure the pressure between two mains and the current in either of them.

For a star connected load we find in a similar manner

$$\cos\phi = \frac{W_1 + W_2 + W_3}{E_1A_1 + E_2A_2 + E_3A_3}.$$

When the load is balanced we have

$$\cos\phi = \frac{W_1 + W_2 + W_3}{\sqrt{3}V_{2.3}A_1} = \frac{W}{\sqrt{3}VA}.$$

When only one wattmeter is available, the following approximate method of finding the power factor is sometimes used.

Wattmeter method of finding $\cos\phi$.

Let the P.D.'s and currents of the balanced load be represented by the lines drawn in Fig. 101. We have made the assumption that the current and P.D. vectors can be represented by lines drawn in one plane. This assumption is true when the currents and potential differences follow the harmonic law.

The P.D. vectors are always in one plane, for

$$v_{1.2} + v_{2.3} + v_{3.1} = 0,$$

and the current vectors are also in one plane since

$$a_1 = i_3 - i_2,$$

but all the vectors would only be in one plane when a linear relation connected every three of them (Chapter VIII).

Let ϕ be the angle between I_3 and $V_{1.2}$ and also between I_2 and $V_{3.1}$. Since the load is balanced, the angle between I_2 and I_3 is 120 degrees and that between I_3 and A_1 is 30 degrees, for A_1 is equal to the resultant of I_3 and $-I_2$.

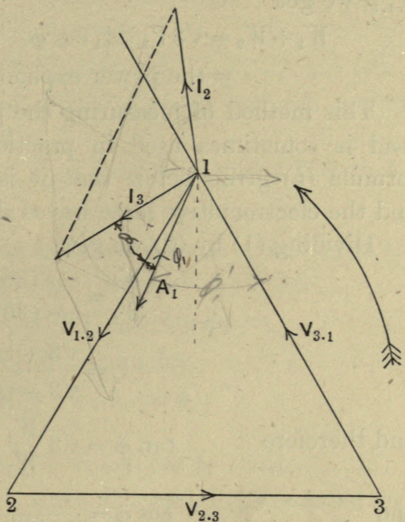

Fig. 101. P.D.'s and currents on a three phase inductive mesh load when balanced.

Put the ampere coil of the wattmeter in the main 1 and connect the volt coil first between 1 and 2 and then between 1 and 3; let W_1 and W_2 be the readings in the two cases, then

$$W_1 = V_{1.2} A_1 \cos \phi_1,$$

where ϕ_1 is the angle between $V_{1.2}$ and A_1 in Fig. 101.

Since the angle between I_3 and $V_{1.2}$ is ϕ and between I_3 and A_1 is 30 degrees, therefore

$$\phi_1 = 30° - \phi.$$

Hence $$W_1 = V_{1.2} A_1 \cos (30° - \phi) \quad \text{.................. (1).}$$

If we now disconnect the terminal of the volt coil from No. 2 main and connect it with No. 3, the reading W_2 on the wattmeter will be

$$W_2 = V_{1.3} A_1 \cos \phi_1',$$

where ϕ_1' is the angle between $V_{1.3}$ and A_1. Now $\phi_1' + \phi_1$ is the angle between $V_{1.2}$ and $V_{1.3}$ and is therefore equal to 60 degrees.

Thus $$W_2 = V_{1.3} A_1 \cos (30° + \phi) \quad \text{.................. (2).}$$

By adding (1) and (2) and noting that $V_{1.2}$ and $V_{2.3}$ both equal $V_{1.3}$, we get

$$W_1 + W_2 = \sqrt{3}\, V_{1.3} A_1 \cos\phi$$

$$= \text{the power expended on the balanced load.}$$

This method of measuring the power expended on a balanced load is sometimes used in practice and it will be seen from formula (*b*) given below that it is applicable when the current and the electromotive force waves do not follow the harmonic law.

Dividing (1) by (2) we get

$$\frac{W_1}{W_2} = \frac{\cos(30° - \phi)}{\cos(30° + \phi)}$$

$$= \frac{\sqrt{3} + \tan\phi}{\sqrt{3} - \tan\phi},$$

and therefore $$\tan\phi = \sqrt{3}\,\frac{W_1 - W_2}{W_1 + W_2},$$

and $$\cos\phi = \frac{W_1 + W_2}{2\,(W_1^2 + W_2^2 - W_1 W_2)^{\frac{1}{2}}}.$$

This method gives the value of $\cos\phi$ by means of two wattmeter readings when the waves follow the harmonic law. In other cases the formula is only approximate.

Phase indicator.

We shall now consider the theory of the phase indicator, which is an instrument to measure the power factor of a balanced load on a polyphase system. The scale of this instrument may be graduated by making its readings equal the phase differences calculated from the readings of a wattmeter, an ammeter and a voltmeter when they are all connected in the proper manner with a balanced load the power factor of which can be varied; and this is the method adopted in practice. It is instructive, however, to consider the theoretical basis for the design of instruments of this class, as this enables us to find out their limitations and shows how some of their defects may be remedied.

Let us take the case of a three phase instrument. We saw on page 236 that the instantaneous value w of the total power taken by a three phase load is given by

$$w = v_1 a_1 + v_2 a_2 + v_3 a_3.$$

If W be the mean value of w, then when the load is balanced, we have, from symmetry,

$$W = 3 V_1 A_1 \cos\phi$$
$$= \sqrt{3}\, V_{1.2} A_1 \cos\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

where $\cos\phi$ is the power factor of the balanced load and ϕ is the phase difference between v_1 and a_1.

We get in a similar manner, from formula (1) page 236, that

$$W = V_{1.2} A_1 \cos\phi_1 + V_{3.2} A_3 \cos\phi_3',$$

where ϕ_1 is the phase difference between $v_{1.2}$ and a_1 and ϕ_3' is the phase difference between $v_{3.2}$ and a_3. Similarly we have

$$W = V_{2.3} A_2 \cos\phi_2 + V_{1.3} A_1 \cos\phi_1'.$$

If the load be balanced, we have, by symmetry,

$$V_{1.3} A_1 \cos\phi_1' = V_{3.2} A_3 \cos\phi_3',$$

and thus we have

$$W = V_{1.2} A_1 \cos\phi_1 + V_{1.3} A_1 \cos\phi_1' \ldots\ldots\ldots\ldots\ldots (b).$$

By equating (a) and (b) we get, since $V_{1.2}$ and $V_{1.3}$ are equal on a balanced load,

$$\cos\phi_1 + \cos\phi_1' = \sqrt{3}\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

Since we always have

$$v_{1.2} + v_{2.3} + v_{3.1} = 0,$$

$V_{1.2}$, $V_{2.3}$ and $V_{3.1}$ form a triangle and when the sides are all equal $V_{1.2}$ and $V_{3.1}$ are inclined to one another at an angle of 120°.

We shall now make the assumption that $v_{1.2}$, $v_{3.1}$ and a_1 follow the harmonic law so that the three vectors $V_{1.2}$, $V_{3.1}$ and A_1 are in one plane. Since ϕ_1 is the angle between $V_{1.2}$ and A_1 and $180° - \phi_1'$ is the angle between $V_{3.1}$ and A_1 and the angle between $V_{1.2}$ and $V_{3.1}$ is 120°, we must have

$$180° - \phi_1' - \phi_1 = 120°,$$

and therefore $$\phi_1 + \phi_1' = 60°.$$

Substituting $60° - \phi_1$ for ϕ_1' in (c) we get

$$\cos\phi_1 + \cos(60° - \phi_1) = \sqrt{3}\cos\phi.$$

Therefore $$2\cos 30° \cos(30° - \phi_1) = \sqrt{3}\cos\phi,$$

and thus and $$\left.\begin{array}{l}\phi_1 = 30° - \phi\\ \phi_1' = 30° + \phi\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

We thus see that if we make the assumption that the currents and the potential differences follow the harmonic law, the angle between $V_{1.2}$ and A_1 is $30° - \phi$, and numerous attempts have been made to construct a phase indicator which will utilise this geometrical property of the vectors.

In the phase indicator described below we have two fixed coils and one moveable coil. The two fixed coils have the same number of turns, and each is put in series with a large non-inductive resistance R. In Fig. 102 let OA and OB be the axes of fixed coils which are connected between the mains 1 and 2 and between the mains 1 and 3 respectively. In practice, it is customary to arrange the coils so that if a current flowing from 1 to 2 produces a magnetic field in the direction OA, a current flowing from 3 to 1 will produce a magnetic field in the direction BO. Since R is large, the current in the fixed coil which has OA for its axis is in phase with $v_{1.2}$ and the current in the other fixed coil is in phase with $v_{3.1}$. Hence, if the time be reckoned from the instant when the potential difference between the mains 1 and 2 is a maximum, the current in the fixed coil between 1 and 2, and therefore the strength of the magnetic field in the direction OA due to this current, is proportional to $\cos \omega t$ and we shall denote it by $h_1 \cos \omega t$. Similarly the strength of the field in the direction OB due to the current in the other fixed coil may be denoted by $-h_1 \cos(\omega t - \psi)$, where ψ is the phase difference between $v_{1.2}$ and $v_{3.1}$.

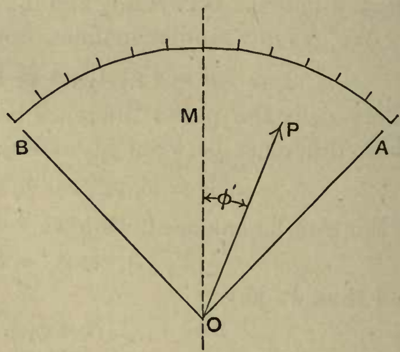

Fig. 102. Phase Indicator.

Let the angle AOB in Fig. 102 be denoted by α, and let OM bisect the angle AOB. Then, if h_x and h_y be the components of the resultant magnetic field due to the currents in the two fixed coils perpendicular to and along OM respectively, we have

$$h_x = h_1 \cos \omega t \sin \frac{\alpha}{2} + h_1 \cos(\omega t - \psi) \sin \frac{\alpha}{2}$$

$$= 2h_1 \sin \frac{\alpha}{2} \cos \frac{\psi}{2} \cos\left(\omega t - \frac{\psi}{2}\right),$$

and
$$h_y = h_1 \cos \omega t \cos \frac{\alpha}{2} - h_1 \cos(\omega t - \psi) \cos \frac{\alpha}{2}$$
$$= -2h_1 \cos \frac{\alpha}{2} \sin \frac{\psi}{2} \sin\left(\omega t - \frac{\psi}{2}\right).$$

We see that when α equals ψ, h_x and h_y are the projections of a line of length $2h_1 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$ or $h_1 \sin \alpha$, which revolves with a constant angular velocity ω. If, therefore, the angle AOB is made equal to ψ, the resultant magnetic field due to the currents in the fixed coils is of constant strength and rotates with uniform angular velocity. Since the indicator is only used to find the power factor on balanced loads, ψ is constant and equals 120 degrees. Let us suppose that the angle AOB is made equal to this, so that we always have a pure rotating field.

The strength h of the magnetic field resolved along OP is given by
$$h = h_x \sin \phi' + h_y \cos \phi'$$
$$= h_1 \sin \alpha \sin\left(\phi' + \frac{\alpha}{2} - \omega t\right),$$
and since α equals $120°$ we get
$$h = \frac{\sqrt{3}}{2} h_1 \cos(\omega t + 30° - \phi') \quad \ldots\ldots\ldots\ldots\ldots(e).$$

Now the moveable coil carries an alternating current which is in phase with a_1. We can therefore suppose that it is replaced by a magnet the polarity of which alternates with the current. If the coil be small, the action of this magnet may be imitated exactly by two small permanent magnets which rotate in opposite directions with equal angular velocities ω and the moments of which are half the maximum moment of the alternating current magnet.

To prove this, consider two small magnets each of moment $\frac{1}{2}H$ rotating side by side in opposite directions with equal angular velocities ω, and let the time be reckoned from the instant when the magnetic fields due to each are pointing in the same direction. Then at the time t, the component field in this direction at unit distance is $H \cos \omega t + H \cos(-\omega t)$, that is, $2H \cos \omega t$, and the component field perpendicular to this direction is
$$\tfrac{1}{2} H \sin \omega t + \tfrac{1}{2} H \sin(-\omega t)$$

or zero (see page 13). We see therefore that they are equivalent to the single alternating current magnet which, at unit distance, produces the field $2H \cos \omega t$ in the direction of its axis.

Now the action of the rotating field on the equivalent permanent magnet which is rotating in the same direction as itself is to produce a couple which acts so as to diminish the angle between this magnet and the direction of the rotating field. The mean couple produced by the rotating field on the equivalent magnet which rotates in the other direction is zero. Thus the moveable coil is in a position of equilibrium when the direction of the rotating field due to the fixed coils coincides with the direction of the equivalent magnet which rotates in the same direction.

Let OP (Fig. 102) be the position of the axis of the moveable coil when in equilibrium. At the instant when OP is the direction of the rotating field, h, which acts along this direction, has its maximum value and both the equivalent permanent magnets are also pointing in this direction. Thus h is in phase with a_1, and therefore from (e) the phase difference between a_1 and $v_{1.2}$ is $30° - \phi'$. But, by the equation (d), this angle is equal to $30° - \phi$ when the load is balanced, and thus ϕ' must be equal to ϕ. If therefore the scale of the instrument be divided into degrees, the cosine of the reading will give the power factor, when the potential differences and the currents follow the harmonic law.

If the axes of the fixed coils be not placed so as to include an angle of 120 degrees, then in general, both the angular velocity and the magnitude of the rotating field due to the fixed coils vary at different instants even when the applied waves are sine shaped. It is found however that the pointer gives a definite reading for a load of given power factor, and so, with the aid of a wattmeter, an ammeter and a voltmeter the scale can be marked.

In Fig. 103 are shown the connections of the Heap phase indicator which is constructed on the above principle. R and R are resistances in series with the fixed coils, and the moveable coil carries the whole current. In the Heap phase indicators for high pressure working the fixed coils are the secondaries of transformers the primaries of which are connected between 2 and 1 and between 1 and 3. The moveable coil sometimes carries only a fraction of the current in the main 1, and sometimes it is the secondary of a transformer whose primary consists of one or two turns placed in series with the main 1. It is thus possible to arrange the indicator in the circuit so that only low pressure wires come to the instrument.

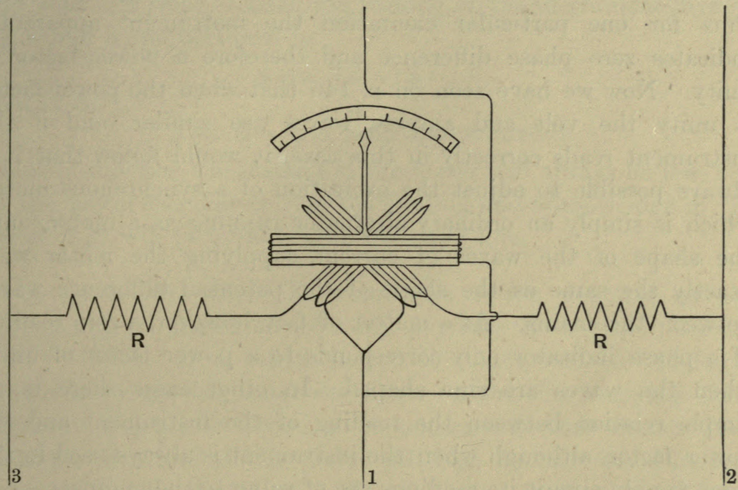

Fig. 103. Connections of Heap Phase Indicator for three phase working.

Let us suppose that the phase indicator is connected in the usual manner with the three mains supplying power to the windings of the armature of a three phase synchronous motor (see Vol. II, Chap. v). Then, it will be proved in Vol. II that, when the potential differences and the currents follow the harmonic law, we can alter the power factor by varying the direct current excitation required by the machine, although the mechanical power given out by the motor is maintained at a constant value. For a particular excitation we can show that the power factor is unity. For excitations less than this the current in a winding lags behind the potential difference applied at the terminals of the winding, and for excitations greater than this it leads the potential difference, the winding of one phase acting like a condenser with a non-inductive resistance in series with it. Thus, as we increase the excitation of the motor, the pointer of the phase indicator will move from one side of the scale to the other, passing through the position where it reads zero, and where consequently the power factor is unity. In practice the waves of potential difference and current are not sine shaped, but the pointer moves in a similar manner from one side of the scale to the other as the excitation varies from a low to a high value, and thus for one particular excitation the instrument apparently indicates zero phase difference and therefore a power factor of unity. Now we have seen on p. 146 that when the power factor is unity the volt and ampere waves are similar, and if the instrument reads correctly in this case, it would follow that it is always possible to adjust the excitation of a synchronous motor, which is simply an ordinary alternator running as a motor, until the shape of the waves of current supplying the motor were exactly the same as the shape of the potential difference waves between the mains. As a matter of fact, however, a zero reading of a phase indicator only corresponds to a power factor of unity when the waves are sine shaped. In other cases there is no simple relation between the reading of the instrument and the power factor, although when the instrument is always used in the same supply circuit its readings are of value to the engineer.

Meter for the wattless current.

If we make the shunt circuit of an electromagnetic wattmeter, which is constructed on the dynamometer principle and has no mutual inductance between its coils in the zero position, very inductive, then the current A_1 in it will lag in phase by nearly ninety degrees behind the applied potential difference V. Such an instrument may be used to measure the wattless current (see page 159). The reading of the instrument will be proportional to $AA_1 \cos \alpha$, where A is the main current and α is the phase difference between A and A_1. If the power factor of the load be $\cos \phi$, A will lag behind V by ϕ degrees and A_1 lags behind V by ninety degrees. When the applied potential difference and the currents follow the harmonic law, the three vectors representing V, A and A_1 lie in one plane (see Chapter VIII), and therefore α is $90° - \phi$. Since A_1 is proportional to V, the reading of the instrument is proportional to $VA \cos(90° - \phi)$, that is, to $VA \sin \phi$. Thus when V is known we can find the wattless current $A \sin \phi$.

If one or more of the waves do not follow the harmonic law, the three vectors form a solid angle, and $\phi+\alpha$ is therefore greater than ninety degrees. Let us suppose that

$$\phi+\alpha=90^\circ+x$$

and that x is small. Then the reading of the instrument divided by V will be proportional to

$$A\cos\{90^\circ-(\phi-x)\},$$

that is, to

$$A\sin\phi-x\,.\,A\cos\phi \text{ approximately.}$$

Thus, in practice, the reading of the meter will always be low, and so the power calculated from the reading and a knowledge of the value of A, will be greater than its true value.

Induction type watt-hour meters.

A watt-hour meter in which the action depends on the induction of eddy currents in a metal disc or drum which is free to rotate, and on the forces produced by moving magnetic fields on the disc carrying these currents, is generally called an induction watt-hour meter. The principle of action of this meter will best be understood by considering the simplest form. We shall suppose that there is an aluminium disc fixed on the spindle and that this disc is free to rotate in the horizontal air-gaps of two C-shaped electromagnets placed side by side at its circumference. The winding of one of these magnets is in series with a choking coil and the combination is connected between the mains. The current in the winding of this magnet, which we shall call the shunt magnet, lags by nearly ninety degrees behind the potential difference applied to the load. The other magnet—the series magnet—is excited by the main current itself or by a current in phase with it.

Let us suppose that the disc is fixed. When currents flow in the windings of both magnets, the alternating magnetic fluxes in the air-gaps will induce eddy currents in the disc. The phases of the tubes or stream lines of current generated in the disc will depend on the resistance and inductance of their paths. The eddy currents produced by the field due to the shunt magnet will be acted on by the magnetic field due to the series magnet and *vice versâ*. The mean values of the electromagnetic attractions and repulsions do not in general balance one another and thus a torque is produced. If the disc rotate, the shape and position of the stream lines relatively to the poles will be altered, and electromotive forces will be set up in the metal of the disc by its motion. As a rule, however, we may neglect the electromotive forces due to the motion of the metal in comparison with the electromotive forces set up by the alternating flux. In practice, even at full load, the disc makes less than one revolution per second. The alternating flux, on the other hand, will, if the frequency of the supply current is fifty, alter its direction one hundred times per second. We may therefore in finding an approximate formula neglect the effects of rotation.

Let i_1 be the strength at a particular instant of one of the tubes of current induced in the disc by the field of the shunt magnet. The torque produced by the action of this tube of current on the series magnet, if we assume that the permeability of the iron is constant, must be of the form $k_1 n_2 I_2' i_1$ where n_2 is the number of turns in the series winding, k_1 is a constant that depends on the relative positions of the tube of eddy current and the series magnet, and I_2' is the instantaneous value of the current exciting the series magnet. Thus, taking the sum of the torques produced by the series magnet on all the tubes of current generated in the disc by the shunt magnet, we get $n_2 I_2' \Sigma k_1 i_1$ for the total torque. Similarly $-n_1 I_1' \Sigma k_1' i_2$ will represent the total torque produced by the shunt magnet on the eddy currents in the disc generated by the series magnet. Thus if g represent the instantaneous value of the resultant torque on the disc, we have

$$g = n_2 I_2' \Sigma k_1 i_1 - n_1 I_1' \Sigma k_1' i_2 \quad \text{..................(1)}.$$

Let us now suppose that the polar faces of the magnets are equal, and that they are similarly situated with respect to the disc so that for every tube of current i_1 generated by one of them there is a corresponding tube of current i_2 generated by the other, and the constant k_1 for a circuit of one set of stream lines equals the constant k_1' for a circuit of the other set. We shall also suppose that the currents follow the harmonic law. We may write, therefore,

$$I_1' = I_1 \cos(\omega t - \alpha), \qquad I_2' = I_2 \cos(\omega t - \beta),$$
$$i_1 = n_1 l_1 I_1 \cos(\omega t - \alpha_1), \qquad i_2 = n_2 l_1 I_2 \cos(\omega t - \beta_1),$$
$$i_1' = n_1 l_2 I_1 \cos(\omega t - \alpha_2), \qquad i_2' = n_2 l_2 I_2 \cos(\omega t - \beta_2),$$

....

The values of l_1, l_2, ..., α_1, β_1, ..., depend on the values of the inductance and resistance of the paths of the eddy currents. Their values also depend on the frequency.

Let the phase difference between the currents in the magnet windings be γ, then we shall have

$$\gamma = \beta - \alpha = \beta_1 - \alpha_1 = \ldots.$$

Let also δ_1 be the phase difference between i_1 and I_1'; then δ_1 will also be equal to the phase difference between i_2 and I_2'. Thus we have

$$\delta_1 = \alpha_1 - \alpha = \beta_1 - \beta.$$

Let also
$$\delta_2 = \alpha_2 - \alpha = \beta_2 - \beta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting in (1) we get

$$g = n_2 n_1 I_2 I_1 \cos(\omega t - \beta)\{k_1 l_1 \cos(\omega t - \alpha_1) + k_2 l_2 \cos(\omega t - \alpha_2) + \ldots\}$$
$$- n_1 n_2 I_1 I_2 \cos(\omega t - \alpha)\{k_1 l_1 \cos(\omega t - \beta_1) + k_2 l_2 \cos(\omega t - \beta_2) + \ldots\}.$$

If G denote the mean torque, we have

$$G = \frac{n_2 n_1 I_1 I_2}{2}\{k_1 l_1 \cos(\alpha_1 - \beta) + k_2 l_2 \cos(\alpha_2 - \beta) + \ldots$$
$$- k_1 l_1 \cos(\beta_1 - \alpha) - k_2 l_2 \cos(\beta_2 - \alpha) + \ldots\}$$
$$= n_1 n_2 I_1 I_2 \left\{k_1 l_1 \sin\frac{\alpha_1 - \alpha + \beta_1 - \beta}{2} \sin\frac{\beta_1 - \alpha_1 + \beta - \alpha}{2} + \ldots\right\}$$
$$= n_1 n_2 I_1 I_2 \sin\gamma\,\{k_1 l_1 \sin\delta_1 + k_2 l_2 \sin\delta_2 + \ldots\}.$$

The values of δ_1, δ_2 ... depend on the conductivity of the metal disc. If the temperature and the frequency remain constant, the expression $k_1 l_1 \sin\delta_1 + k_2 l_2 \sin\delta_2 + \ldots$ will be a constant. We thus see that, when we make the sine curve assumption, G will be proportional to $I_1 I_2 \sin\gamma$.

Now, since we can neglect the actions of the small eddy currents, I_1 is proportional to the voltage V applied to the load and I_2 is proportional to the effective value A of the load current. The accelerating torque G is therefore proportional to $VA\sin\gamma$. The angle γ is the phase difference between the currents in the windings of the shunt and series magnets. On a non-inductive load it is nearly ninety degrees; and since we are assuming that all the currents follow the harmonic law, γ will equal $90^\circ - \phi$ when $\cos\phi$ is the power factor of the load. It therefore follows that the resultant accelerating torque is proportional to $VA\cos\phi$, that is, to the power being expended on the load.

The retarding torque is produced by the eddy currents generated in the rotating disc by the field due to permanent magnets. These magnets are generally C-shaped, and the circumference of the disc rotates in the air-gap. The retarding torque will therefore be proportional to the angular velocity of the spindle. It is to be noted that, since the eddy currents due to the series and shunt magnets are alternating, the mean effect of the permanent magnets on them is zero. The shunt and series magnets also have on the average no effect on the eddy currents due to the permanent magnets. When the motion is steady, the accelerating torque is equal to the retarding torque, and thus the power being expended on the load is proportional to the angular velocity of the spindle. The spindle is connected with the counting mechanism by worm gearing and so a record is made of the energy expended.

Instead of a disc, a light hollow cylinder of aluminium is sometimes used, the shunt and series magnets being placed near to one another and facing the circumference of the cylinder. As before the retarding torque is produced by the eddy currents due to permanent magnets. The theory of the action of this instrument is practically identical with that of the disc meter.

In meters which have an aluminium disc, a compound magnet made up of ᗰ-shaped iron stampings is often used instead of two separate magnets for the shunt and series windings. The shunt coil is wound on the middle limb and the series coils on one or both of the outer limbs. The compound magnet is placed near the circumference of the aluminium disc, and strips of iron are fixed underneath the disc so as to reduce the reluctance of the magnetic circuits of the compound magnet. When there is a current in the series coil, the mean values of the magnetic fluxes in the two outer limbs are unequal, as the magnetising force of the shunt coil increases the resultant magnetising force in one limb and diminishes it in the other. This effect alters the ratio of the angular velocity of the disc to the effective current in the series coils, and tends to make the meter read inaccurately. It can be neutralised by placing a few turns of series winding on the middle limb.

This type of meter can easily be adapted to register the energy expended in a two or three phase load. All that we need do is to apply the two wattmeter method (see Chapters XI and XII) and combine the two watt-hour meters into one instrument. Two ⊓-shaped magnets are arranged in this case to act on the same aluminium disc. They are placed facing the top surface on opposite sides of the centre of the disc. If they are suitably wound and connected with the mains as in the two wattmeter method, the sum of the accelerating torques, that is, the resultant torque, will be proportional to the total power being expended on the load.

In other meters of the induction type, we have a metal disc placed in a rotating magnetic field. The principle of these meters will easily be understood from the theory of the induction motor developed in Volume II.

A serious objection to the use of meters, the action of which depends on the generation of eddy currents in masses of metal, is their large temperature coefficient. The change of the resistivity of copper and aluminium, the metals usually employed, is about 0·4 per cent. per degree centigrade. In the expression given above for the accelerating torque it will be seen that $l_1, l_2 \ldots$ all depend on the resistance of the paths of the eddy currents. They therefore vary with the temperature. Unless some compensating device be employed, the meter will only read correctly at a particular temperature. It will also only read correctly at a given frequency.

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CHAPTER XIV.

Rotating magnetic fields. Resultant field due to n equal and symmetrically placed poles supplied with n phase currents. Equal poles unevenly spaced; phase differences of magnetic vectors equal to their angular distances apart. General case. Properties of rotating and alternating magnetic fields. Magnetic field in the air-gap of polyphase machines. Rotating field in the air-gap of a polyphase induction motor. Gliding magnetic fields. Rotating magnetic fields when the currents are not sine shaped. Rotating magnetic field producing a constant effective E.M.F. in a search coil placed with its plane perpendicular to the field. Extension to three phase theory. Arnò's phase indicator. References.

MAGNETIC forces are compounded by the parallelogram law, and hence we may apply statical constructions in order to find their resultant. For example, suppose that the magnetic forces at the point O (Fig. 104) are represented in magnitude and direction by the lines OA_1, OA_2, OA_3 and OA_4. Then, if G be the centre

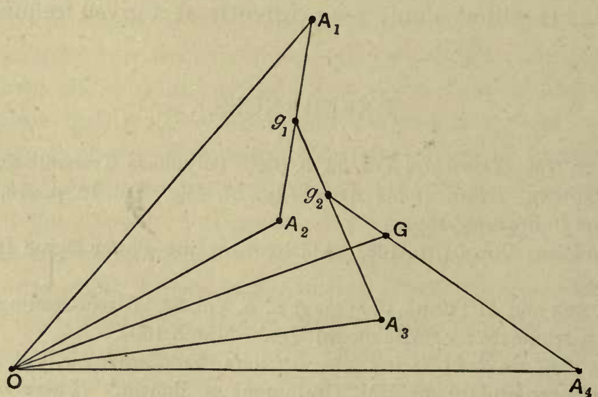

Fig. 104. Resultant of OA_1, OA_2, OA_3 and $OA_4 = 4 . OG$.

of gravity of equal masses placed at A_1, A_2, A_3 and A_4, the resultant of the forces will be represented in magnitude by $4 . OG$, and in direction by OG. If there had been n forces, then the resultant would have been equal to $n . OG$.

To find G, we bisect $A_1 A_2$ in g_1, and then make

$$g_1 g_2 = \tfrac{1}{3} g_1 A_3, \quad g_2 g_3 = \tfrac{1}{4} g_2 A_4, \text{ etc.}$$

This construction is very simple in practice. It is easy to see that it is true whether the lines are in one plane or not. The necessary and sufficient condition that the forces are in equilibrium is that the centre of gravity of equal masses placed at the extremities of the lines representing the forces coincides with O. We have seen that this theorem also holds for alternating current vectors, and we have already made use of it in the chapters on two and three phase theory.

Rotating magnetic fields.

In a rotating magnetic field the direction of the magnetic force is continually revolving. If we move a strong permanent magnet round a small compass needle, keeping the same end of the magnet always pointing to the needle, we produce a rotating magnetic field at the centre of the small compass. The needle at any instant points out the direction of the magnetic field, and its angular velocity measures the velocity of rotation of the field.

If the core of an electromagnet be made up of thin strips of soft iron insulated from one another by means of shellac varnish, or by paper pasted on one side of each strip, we get an alternating current magnet. When the windings of such a magnet are supplied with alternating currents, the polarity of the ends of the magnet alternates with the same frequency as the alternating currents and a varying magnetic field is produced in the neighbourhood. The magnet is usually made with a straight core. The iron in the core needs to be laminated, for otherwise its temperature would rise excessively owing to the heat developed by the eddy currents that would be induced in it. A simple way of making such a magnet is to take for the core a cylindrical bundle of insulated iron wires, the lengths of which are parallel to the axis of the cylinder, and then to wind insulated copper wire round the cylinder to carry the alternating current required to magnetise the core.

Now place two alternating current magnets (Fig. 105) at right angles to one another, with axes pointing in the directions AOA' and BOB' respectively. If the magnetic force at O due to the first be $H \cos \omega t$ and that due to the second be

$$H \cos\left(\omega t - \frac{\pi}{2}\right),$$

and if (Fig. 105)

$$OQ = H \cos \omega t,$$

and

$$OP = H \cos\left(\omega t - \frac{\pi}{2}\right) = H \sin \omega t,$$

then, if OR be the resultant of OP and OQ, we have

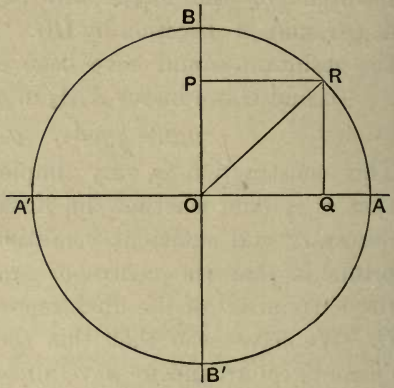

Fig. 105. Pure Rotating Field produced by two alternating fields in quadrature.

$$OR^2 = OP^2 + OQ^2 = H^2;$$

and therefore $$OR = H = \text{a constant},$$

and the angle OR makes with OA will be ωt, so that OR will rotate with constant angular velocity ω. In this case, then, we obtain a pure rotating magnetic field, that is, one which is not only constant in magnitude but rotates with constant angular velocity. If we used two solenoids without iron which were supplied with sine shaped current waves from a two phase machine, we could produce this field. The rotating fields produced in practice, although often constant in magnitude, rarely rotate with constant angular velocity. For the present we will consider the case of pure rotating fields, and consequently we make the supposition that the alternating currents inducing the fields are sine shaped.

Resultant field due to n equal and symmetrically placed poles supplied with n phase currents.

Consider the case of the field produced by n equal poles arranged evenly round a circle, the currents producing the poles differing in phase by $\frac{360}{n}$ degrees.

Let the strength of the field at O, due to the pole P_1, be $H \cos \omega t$ in the direction Op_1.

Similarly let the field at O, due to P_2, be $H\cos\left(\omega t - \frac{360°}{n}\right)$ in the direction Op_2, and so on for the remaining poles. With centre O and radius equal to H describe a circle, and let a circle of radius $\frac{H}{2}$ roll round this circle $\frac{\omega}{2\pi}$ times a second, the centre of the moving circle lying on OP_1 when t is zero. Then the intercepts

$$Op_1,\ Op_2, \ldots\ldots$$

will represent $H\cos\omega t$,

$$H\cos\left(\omega t - \frac{360°}{n}\right)\ldots\ldots,$$

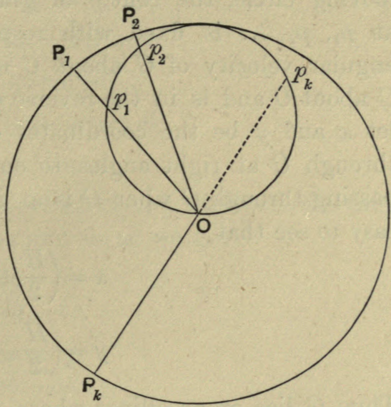

Fig. 106. Resultant Field of n phase currents equals $\frac{1}{2}nH$.

that is the values of the component strengths of the fields. It is easily shown that p_1, p_2, ... are fixed points on the moving circle. Since the angles p_1Op_2, p_2Op_3, ... are all equal, $p_1p_2 \ldots p_n$ is a regular polygon. If C be the centre of the rolling circle, C is always the centre of gravity of equal masses placed at p_1, p_2, ... Therefore the resultant magnetic force is represented in magnitude and direction by $n\,.\,OC$. Hence the resulting field is a pure rotating one of magnitude $\frac{1}{2}nH$.

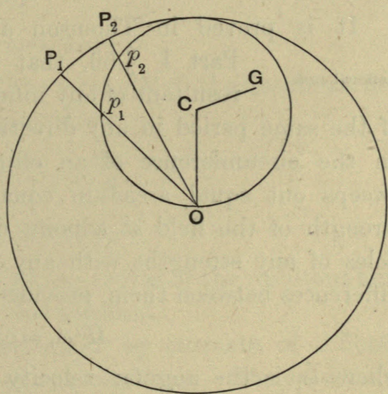

Fig. 107. Elliptic Field produced by poles unevenly spaced. $H^2\omega_1 = \text{constant}$.

Equal poles unevenly spaced; phase differences of magnetic vectors equal to their angular distances apart.

As in the last case, suppose that a circle rolls inside another of double its diameter. Then p_1p_2, p_2p_3, ... (Fig. 107) subtend constant angles at the circumference of the circle, and therefore the polygon $p_1 p_2 \ldots p_n$ is of constant size. Since p_1, p_2, ... are fixed relatively to the moving circle, the centre of gravity G of equal masses placed at p_1, p_2, ... is fixed with respect to the rolling circle. The angular velocity of G about C is double the angular velocity of C about O and is in the reverse direction. Let CG equal c, and let x and y be the coordinates of G with respect to two axes through O at right angles to one another. Then, the axis of x passing through G when G is at its greatest distance from O, it is easy to see that

$$x = \left(\frac{H}{2} + c\right) \cos \omega t,$$

$$y = \left(\frac{H}{2} - c\right) \sin \omega t.$$

Thus G lies on an ellipse whose centre is O and axes $H + 2c$ and $H - 2c$ respectively. Since the field is represented in direction and magnitude by $n \,.\, OG$, we see that we do not get a pure rotating field in this case. If ω_1 be the angular velocity of OG about O, then

$$\omega_1 \,.\, OG = \frac{dy}{dt} \cdot \frac{x}{OG} - \frac{dx}{dt} \cdot \frac{y}{OG},$$

and thus $$OG^2 \,.\, \omega_1 = \left(\frac{H^2}{4} - c^2\right) \omega.$$

General case.

It is proved in Thomson and Tait's *Natural Philosophy*, Part I, § 66, that a point, whose motion is the resultant of any number of simple harmonic motions of the same period in any directions and with any phases, moves on the circumference of an ellipse, and that its radius vector sweeps out equal areas in equal times. Hence, if H be the strength of the field at a point due to any number of magnetic poles of any strengths with any directions, and with any phase differences between them, provided they follow the sine law, then

$$H^2 \omega_1 = \text{constant},$$

where ω_1 is the angular velocity with which the resultant field revolves.

The important case in practice is when the vectors are all in one plane. Let us suppose that the direction of the alternating field $H_1 \cos(\omega t - \alpha_1)$ makes an angle ψ_1 with the axis of x. Then, if x and y be the coordinates of the extremity of the vector representing the resultant field, we have

$$x = \Sigma\, H \cos(\omega t - \alpha) \cos \psi$$

$$= a \cos \omega t + b \sin \omega t,$$

and
$$y = \Sigma\, H \cos(\omega t - \alpha) \sin \psi$$

$$= c \cos \omega t + d \sin \omega t,$$

where a, b, c and d are quantities which do not vary with the time. Solving these equations for $\cos \omega t$ and $\sin \omega t$ we get

$$\cos \omega t = \frac{dx - by}{da - bc},$$

and
$$\sin \omega t = \frac{cx - ay}{cb - ad}.$$

Hence, we get

$$(dx - by)^2 + (cx - ay)^2 = (ad - bc)^2,$$

which shows that the locus of the extremity of the vector representing the resultant field is an ellipse.

In the special case when

$$a = b \text{ and } c = d$$

the ellipse becomes a straight line, and thus the resultant field is purely alternating. Again, when

$$a = \pm d \text{ and } b = \mp c$$

the ellipse becomes a circle and the strength of the resultant field is constant at every instant. Also, since

$$r^2 \frac{d\theta}{dt} = x \frac{dy}{dt} - y \frac{dx}{dt},$$

therefore
$$r^2 . \omega_1 = \omega (ad - bc),$$

where ω_1 is the angular velocity of the vector of the resultant field; and thus, if r is constant, ω_1 is also constant.

If the rotating field be produced by currents of different frequencies, we get all manner of varying fields. The curves that would be described by the extremity of the vector representing the instantaneous value of the resultant magnetic force in some of these cases are given in treatises on the theory of sound under the head of Lissajous's figures, and various mechanical devices are described for drawing them.

Properties of rotating and alternating magnetic fields.

We shall now consider a few of the properties of rotating and alternating magnetic fields. By a pure rotating field we mean one whose strength and angular velocity are both constant, and by an alternating field we mean one whose direction is constant but whose strength is a periodic function of the time obeying the sine law. When we consider more than one field, all the fields will be supposed to be parallel to one plane and the frequency both of the rotations of the rotating fields and of the alternations of the alternating fields will be supposed to have the same value. We will suppose the fields represented by rotating and alternating vectors, which may be drawn in one plane. The following theorems will be found useful in practice.

(1) *Two vectors rotating in the same direction in one plane are equivalent to a single rotating vector.*

Let Op and Oq be the two vectors. Since they rotate with the same angular velocity, the angle pOq remains constant. Construct a parallelogram on Op and Oq as adjacent sides, and let OR be the diagonal. This represents the resultant field, which will obviously rotate with the same angular velocity as Op and Oq. Hence two vectors rotating in the same direction compound into a single rotating vector.

(2) *Two equal vectors rotating in opposite directions in one plane are equivalent to a single alternating vector, and conversely a single alternating vector is equivalent to two equal vectors rotating in opposite directions.*

Let Op and Oq be the two equal vectors. At any instant their resultant is $2 . Or$, where r is the middle point of pq. Also since Or is perpendicular to pq, and Op and Oq are rotating with equal angular velocities in opposite directions, Or is fixed in direction. Thus two equal vectors rotating in opposite directions compound into a pure alternating vector the amplitude of which equals $2 . Op$. Similarly an alternating vector whose amplitude is Or may be replaced by two equal rotating vectors whose magnitudes are each equal to $\frac{1}{2} . Or$.

(3) *Two unequal vectors rotating in opposite directions are equivalent to an alternating vector and a rotating vector, and, conversely, an alternating vector and a rotating vector are equivalent to two unequal vectors rotating in opposite directions.*

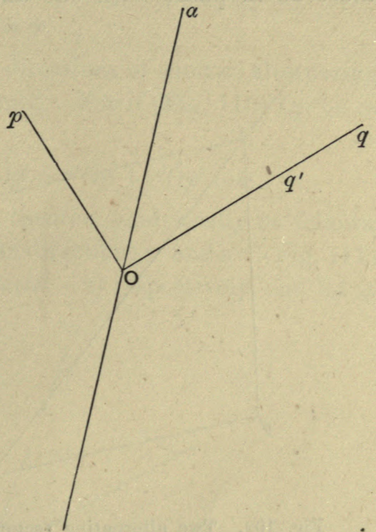

Fig. 108. Two unequal vectors rotating in opposite directions are equivalent to an alternating and a rotating vector.

Let Op and Oq be the magnitudes of the two vectors. Make Oq' equal to Op. Then, since Oq is equivalent to the sum of two rotating fields, in phase with one another, whose magnitudes are Oq' and $q'q$ respectively, and since by (2) Op and Oq' compound into a vector alternating along the line Oa (Fig. 108), which bisects the angle pOq, and having the amplitude $2 . Op$, we see that the given vectors compound into an alternating vector of amplitude $2 . Op$ and a rotating vector whose magnitude is $Oq - Op$.

(4) *Two unequal alternating vectors can, in general, be replaced by a simple alternating vector and a rotating vector.*

By (2) we can replace the alternating vectors whose directions are Or and Or' (Fig. 109) by four rotating vectors Op, Oq and Op', Oq' of which Op and Op' rotate in one direction, and Oq and Oq' in the other, and where Op is half the maximum magnitude of Or and Op' is half the maximum magnitude of Or'. Hence, by (1), we can replace Op and Op' by OP, and Oq and Oq' by OQ where OP and OQ are vectors rotating in opposite directions. If OP and OQ are equal to one another, the resultant field is, by (2), a purely alternating one. If either OP or OQ be zero, the field is a purely rotating one. In all other cases we see, by (3), that it can be represented by an alternating vector and a rotating vector.

When OP equals OQ the resultant field is purely alternating, and it is easy to show that in this case the two component fields must be in phase with one another. To get a purely rotating

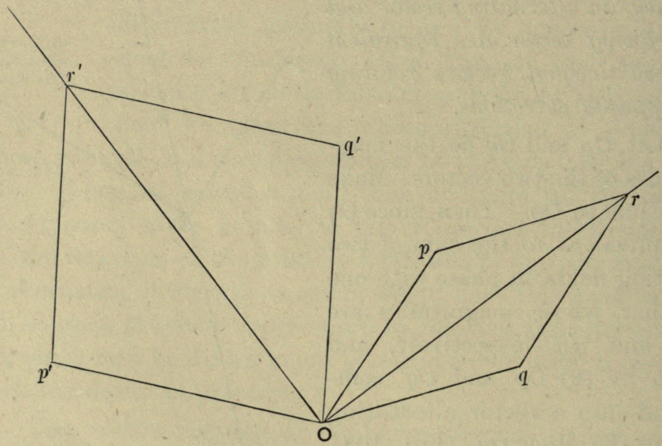

Fig. 109. Two alternating vectors can be replaced by an alternating and a rotating vector.

field we must have either OP or OQ equal to zero. Hence either $Op = Op'$ or $Oq = Oq'$; in either case the amplitudes of the two alternating fields must be equal in magnitude. In the first case (Fig. 110) Op and Op' must be in the same straight line but

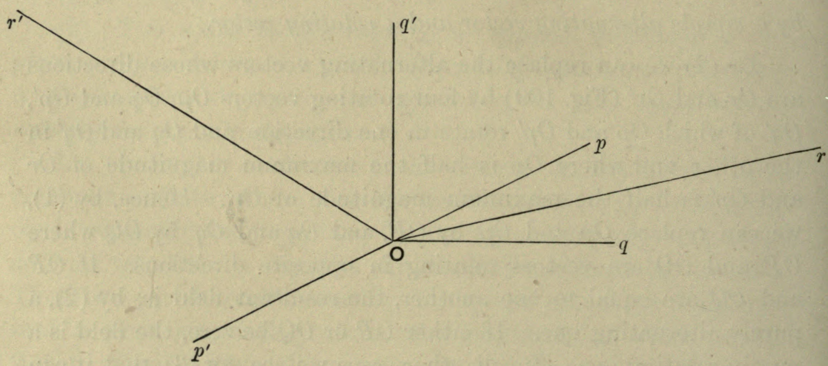

Fig. 110. Op and Op' are in the same line, when the alternating vectors produce a pure rotating field.

pointing in opposite ways, so that the angle between Op and Op' is π.

Hence
$$\phi + \alpha = \pi,$$
where ϕ is the angle between the directions of the two alternating vectors and α is their phase difference. For if (Fig. 110) Op be in the same line as Op', then

$$\phi + \alpha = r\hat{O}r' - r\hat{O}p + r'\hat{O}p' = r'\hat{O}p + r'\hat{O}p' = \pi.$$

We can prove this important theorem analytically as follows. Let the strengths of the fields in the directions sr and $s'r'$ (Fig. 111) be given by $H_1 \cos \omega t$ and $H_2 \cos(\omega t - \alpha)$ respectively, and let ϕ

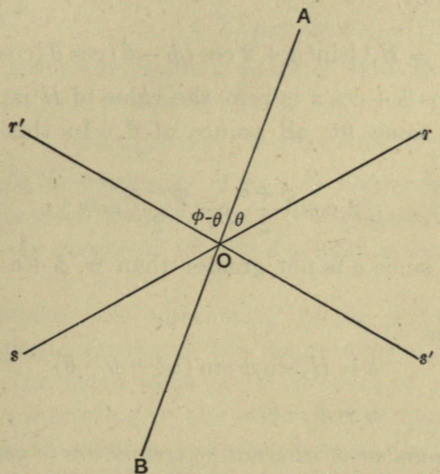

Fig. 111. When the angle rOr' is the supplement of the phase difference between the fields alternating in the directions sr and $s'r'$ we get a pure rotating field.

be the angle rOr'. If sr make an angle θ with AB (Fig. 111) the magnetic force h along OA at the time t is given by

$$h = H_1 \cos \omega t \cos \theta + H_2 \cos(\omega t - \alpha) \cos(\phi - \theta)$$
$$= \{H_1 \cos \theta + H_2 \cos \alpha \cos(\phi - \theta)\} \cos \omega t + H_2 \sin \alpha \cos(\phi - \theta) \sin \omega t$$
$$= H \sin(\omega t + \gamma),$$

where

$$H^2 = H_1^2 \cos^2 \theta + H_2^2 \cos^2(\phi - \theta) + 2H_1 H_2 \cos \alpha \cos \theta \cos(\phi - \theta),$$

and
$$\tan \gamma = \frac{H_1 \cos \theta + H_2 \cos \alpha \cos(\phi - \theta)}{H_2 \sin \alpha \cos(\phi - \theta)}.$$

Now if h be independent of θ, that is, if the amplitude of the alternating field in every direction at O (Fig. 111) be the same, the value of H must be the same when θ is $\frac{\pi}{2}$ and when it is $\phi - \frac{\pi}{2}$. Hence we must have H_1 equal to H_2 and

$$H^2 = H_1^2 \{\cos^2 \theta + \cos^2 (\phi - \theta) + 2 \cos \alpha \cos \theta \cos (\phi - \theta)\}.$$

Now $$\cos^2 \theta = \sin^2 \phi + \cos^2 \theta - \sin^2 \phi$$
$$= \sin^2 \phi + \cos (\phi + \theta) \cos (\phi - \theta).$$

Therefore $$H^2 = H_1^2 [\sin^2 \phi + \cos (\phi - \theta) \{\cos (\phi + \theta) + \cos (\phi - \theta) + 2 \cos \alpha \cos \theta\}]$$
$$= H_1^2 \{\sin^2 \phi + 2 \cos (\phi - \theta) \cos \theta (\cos \phi + \cos \alpha)\}.$$

Thus, when $\cos \phi + \cos \alpha$ is zero, the value of H is $H_1 \sin \phi$, and is therefore the same for all values of θ. In this case we must have

$$2 \cos \frac{\phi + \alpha}{2} \cos \frac{\phi - \alpha}{2} = 0,$$

and therefore, since α is not greater than π, $\phi + \alpha$ must be equal to π.

In this case

$$h = H_1 \sin \phi \sin (\omega t + \phi - \theta)$$

and $$\gamma = \phi - \theta.$$

(5) *Any number of alternating vectors are in general equivalent to a simple alternating vector and a rotating vector.*

We can replace every alternating vector of amplitude Or by two equal vectors Op and Oq rotating in opposite directions. All the component vectors like Op rotating in the same direction can be replaced by a single rotating vector OP. Similarly the Oq components compound into OQ. Hence the theorem follows from (3).

When OP and OQ are equal, the resultant field is a purely alternating one. When either OP or OQ is zero, the resultant field is a purely rotating one. If OP is zero, the resultant of all the Op components is zero, and thus a closed polygon can be constructed whose sides are equal and parallel to all the Op components, and a similar theorem holds when OQ is zero.

(6) *If* $p_1, p_2, \ldots$ *and* $q_1, q_2, \ldots$ *be any vectors and* $\phi_{r.s}$ *be the angle between* p_r *and* q_s, *then*

$$\Sigma\Sigma p_r q_s \cos\phi_{r.s} = PQ\cos\Phi,$$

where P *is the resultant of all the* p *vectors,* Q *is the resultant of all the* q *vectors, and* Φ *is the angle between* P *and* Q.

Resolve all the p vectors along q_1, then by projections

$$p_1\cos\phi_{1.1} + p_2\cos\phi_{2.1} + \ldots + p_n\cos\phi_{n.1} = P\cos\Psi_1.$$

Hence $$q_1\Sigma p_r\cos\phi_{r.1} = Pq_1\cos\Psi_1.$$

Similarly $$q_2\Sigma p_r\cos\phi_{r.2} = Pq_2\cos\Psi_2$$

....

Therefore $$\Sigma\Sigma p_r q_s\cos\phi_{r.s} = P\Sigma q\cos\Psi.$$

But $\Sigma q\cos\Psi$ is the sum of the projections of all the q components upon P, and it therefore equals $Q\cos\Phi$.

Therefore $$\Sigma\Sigma p_r q_s\cos\phi_{r.s} = PQ\cos\Phi.$$

(7) *If* p *be the amplitude of an alternating vector and* q *be a rotating one, the mean value of* $pq\cos\phi$ *is* $\frac{1}{2}pq\cos\delta$, *where* δ *is the angle between the directions of* p *and* q *when* p *has its maximum value.*

We can replace the alternating vector p by two vectors, rotating in opposite directions, the magnitudes of which are each equal to $\frac{1}{2}p$. The mean value of the product of q and the component vector $\frac{1}{2}p$ rotating in the same direction and the cosine of the angle between them is $\frac{1}{2}pq\cos\delta$, because the angle between them is always equal to δ. The mean value of the corresponding product for the other vector is zero, since, for a complete revolution, the mean value of $\cos(\omega t + \delta)$ is zero.

(8) *If* p *and* q *be two alternating vectors whose directions are inclined to one another at an angle* ϕ, *then the mean value of* $pq\cos\phi$ *is*

$$\frac{pq}{4}\{\cos(\phi + \beta - \alpha) + \cos(\phi + \alpha - \beta)\},$$

where $\alpha - \beta$ *is the phase difference between the two vectors.*

The angle between p and q is always ϕ, so that we have only to find the mean value of the instantaneous value of the product of the two vectors. This can be done by (7) for we can replace p by two vectors each equal to $\frac{1}{2}p$ rotating in opposite directions.

The mean value of one of these components multiplied by q and the cosine of the angle between them is by the preceding theorem

$$\frac{pq}{4}\cos\delta,$$

and δ equals $\phi+\beta-\alpha$ or $\phi+\alpha-\beta$ depending on which component we take. The sum of the two will obviously give the mean value of $pq\cos\phi$. This mean value, by adding the cosines, may be written in the shape

$$\frac{pq}{2}\cos\phi\cos(\alpha-\beta).$$

(9) *If p and q be vectors which rotate with different angular velocities, the mean value of $pq\cos\phi$ is zero.*

Suppose that they are rotating in the same direction with angular velocities ω_1 and ω_2; then in the time t where $(\omega_1-\omega_2)t$ equals 2π, the angle ϕ will have increased from 0 to 2π, and therefore its mean value taken over the time t will be zero. Hence, if the mean value be taken over a time t_1 which is large compared with t, it will be zero if t_1 be a multiple of t and will be very small in other cases.

If the vectors rotate in opposite directions, then the time t which the angle ϕ takes to increase from 0 to 2π is given by

$$(\omega_1+\omega_2)t=2\pi$$

and the mean value over a time t_1, large compared with t, will be zero or very nearly zero.

Magnetic field in the air-gap of polyphase machines.

When we are dealing with the magnetic fields in the air-gaps between the rotating and the stationary parts (the rotor and the stator) of several kinds of polyphase machines, a modification of the above method becomes necessary. The following way of treating the problem is due to A. Potier.

In the case considered, we have a hollow laminated cylinder of soft iron with slots along the interior, parallel to the axis, which carry evenly distributed windings for the polyphase currents. When a current flows in one of these windings, it divides the interior into p segments of North and p segments of South polarity. If l be the breadth of one of these segments, then

$2pl$ is the inner circumference of the cylinder. We shall suppose that the next winding is displaced from the first by a distance $\frac{l}{q}$ on the circumference of the cylinder, where q is the number of phases. The rotor consists of a laminated iron cylinder axial with the stator and suitably wound with copper conductors. Like the stator it will have $2p$ poles, but the phases of the currents in it may have any values. The air-gap between the two cylinders may be as small as half a millimetre.

Rotating field in the air-gap of a polyphase induction motor.

Let us consider the case of a polyphase induction motor (Vol. II, Chap. XII). We shall suppose that the windings of the stator are connected with the polyphase supply mains, and that the windings of the rotor consist of closed coils so that the currents in them are due to induction only. When this kind of motor is running, the frequency of the alternating currents in the rotor is small when compared with the frequency of the currents in the stator. We will for the present neglect the magnetising effects of the rotor currents.

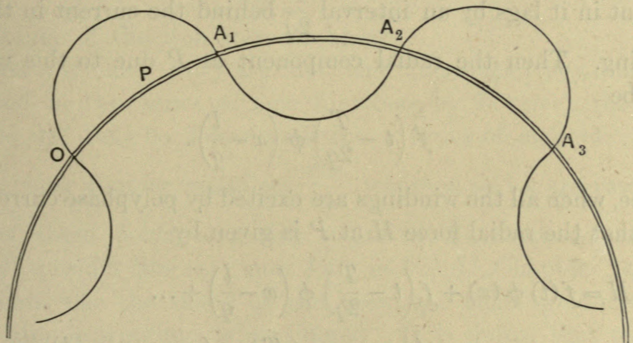

Fig. 112. Field in the air-gap of a polyphase induction motor.

Take some point O (Fig. 112) on the inner surface of the stator and make

$$OA_1 = A_1A_2 = A_2A_3 = \ldots = l,$$

where the points A_1, A_2, $A_3 \ldots$ are on the circumference and the distances between them are measured along the circumference.

If the radial magnetic force at O be zero at the instant when t is zero, the magnitude of the first harmonic of the radial component of the field may be indicated by the wavy line in the figure, the force being positive from O to A_1, negative from A_1 to A_2, etc.

Consider the magnetic force perpendicular to the surface of the stator at a point P, where OP equals x, due to a single alternating current in one of the windings. We may evidently write this in the form $f(t)\,\phi(x)$ where $f(t)$ is some function of the time, and $\phi(x)$ is a function depending on the shape of the rotor and the stator, the method of winding, the permeability of the iron, the air-gap, etc. These functions are alternating periodic functions of the time and space respectively; we have, for example,

$$f(t) = -f\left(t + \frac{T}{2}\right) = f(t+T) = \ldots$$

and
$$\phi(x) = -\phi(x+l) = \phi(x+2l) = \text{etc.}$$

Suppose that the second winding is similar to the first and that it is displaced to the right to a distance $\frac{l}{q}$, and that the current in it lags by an interval $\frac{T}{2q}$ behind the current in the first winding. Then the radial component at P due to this winding will be

$$f\left(t - \frac{T}{2q}\right)\phi\left(x - \frac{l}{q}\right).$$

Hence, when all the windings are excited by polyphase currents, we find that the radial force H at P is given by

$$H = f(t)\,\phi(x) + f\left(t - \frac{T}{2q}\right)\phi\left(x - \frac{l}{q}\right) + \ldots$$
$$+ f\left\{t - (q-1)\frac{T}{2q}\right\}\phi\left\{x - (q-1)\frac{l}{q}\right\} \ldots\ldots(1).$$

It is easily seen that, when t is increased by $\frac{T}{2q}$ and x by $\frac{l}{q}$, the value of H is not altered. If, therefore, we only examine the field at intervals of time $\frac{T}{2q}$, it will appear to glide round the air-gap, without altering in shape, with a linear velocity $\frac{2l}{T}$. If $2p$ be the number of poles and r the radius of the rotor, $2\pi r = 2pl$, and hence the angular velocity of the field is $\frac{2l}{Tr}$ or $\frac{\omega}{p}$, where ω is 2π times the frequency of the alternating current.

If $\qquad f(t) = B \sin \omega t$ and $\phi(x) = C \sin \pi \frac{x}{l}$,

then, substituting in (1) and summing by the ordinary trigonometrical formula, we find

$$H = \frac{q}{2} BC \cos \left(\omega t - \pi \frac{x}{l} \right) \quad \text{..................(2).}$$

Hence, when the magnetic force due to the current follows the harmonic law and when the distribution of the flux round the air-gap is a sine function of the space, the resultant radial field is sine shaped and glides round the air-gap with constant speed. In this case also, if A be the effective current in each phase, the maximum value of the resultant magnetic field would equal that produced by a current $\frac{q}{2} A$ in one phase, provided that the permeability of the iron were constant.

When $f(t)$ and $\phi(x)$ are not sine functions, they may be developed in two series of sine functions by Fourier's theorem. The field will thus be decomposed into a series of magnetic fields rotating with angular velocities $\frac{m\omega}{np}$, where m and n are integers. Some of these fields, also, may rotate in the reverse direction. We shall consider this case more fully in Vol. II, Chapters XII and XIV, where also the shape of the magnetic fields produced by various simple windings is considered. It is shown that the induced currents in the rotor windings produce a magnetic field which rotates in space with the same angular velocity as the magnetic field due to the polyphase currents in the stator windings. The magnetic field which is the resultant of the fields due to the currents in the stator and rotor windings respectively must rotate in space with the same angular velocity as its two components. Thus when we take into account the rotor currents, we still have in the air-gap a magnetic field in which the distribution of the magnetic flux is always exactly the same at intervals of time which differ by multiples of $\frac{T}{2q}$. At any instant also we have p segments of North and p segments of South polarity. This kind of magnetic field we shall call a gliding magnetic field.

If one phase only of the stator windings be excited and the rotor be stationary, then we have as before p segments of North and p segments of South polarity. The polarity of these segments alternates with the frequency of the supply current, but the field is a stationary one, the lines dividing the segments being fixed in position and the magnetic force along these lines being always zero. When we are discussing the magnetic field in the air-gap of an alternating current machine, we shall refer to a field of this kind as an alternating field. Unlike the alternating fields considered earlier in the chapter, the amplitude of the magnetic force is different at different points of the field.

Gliding magnetic fields.

We will now consider what happens when gliding fields which follow the harmonic law are superposed on one another.

(1) *Two fields gliding in the same direction. Time lag but no space lag.*

In this case

$$H = H_1 \sin\left(\omega t - \frac{\pi x}{l}\right) + H_2 \sin\left(\omega \overline{t - \alpha} - \frac{\pi x}{l}\right)$$

$$= R \sin\left(\omega \overline{t - \beta} - \pi \frac{x}{l}\right)$$

where $R^2 = H_1^2 + H_2^2 + 2H_1H_2 \cos\omega\alpha$

and $\tan\omega\beta = \frac{H_2 \sin\omega\alpha}{H_1 + H_2\cos\omega\alpha}$.

Hence the resultant field glides in the same direction with the same velocity, and its magnitude and phase are given by the parallelogram construction.

(2) *Two fields gliding in the same direction. Space lag but no time lag.*

We have

$$H = H_1 \sin\left(\omega t - \frac{\pi x}{l}\right) + H_2 \sin\left(\omega t - \pi\frac{x-a}{l}\right)$$

$$= R \sin\left(\omega t - \pi\frac{x-b}{l}\right)$$

where $$R^2 = H_1^2 + H_2^2 + 2H_1H_2\cos\frac{\pi a}{l}$$

and $$\tan\frac{\pi b}{l} = \frac{H_2\sin\frac{\pi a}{l}}{H_1 + H_2\cos\frac{\pi a}{l}}.$$

The resultant field therefore glides in the same direction with the same velocity, and its magnitude and phase can be got graphically by the parallelogram construction.

(3) *Two equal fields gliding in opposite directions are equivalent to an alternating field.*

In this case,

$$H = H_1 \sin\left(\omega t - \pi\frac{x}{l} + \alpha_1\right) + H_1 \sin\left(\omega t + \pi\frac{x}{l} + \alpha_2\right)$$

$$= 2H_1 \sin\left(\omega t + \frac{\alpha_1+\alpha_2}{2}\right)\cos\left(\pi\frac{x}{l} + \frac{\alpha_2-\alpha_1}{2}\right).$$

The resultant field is therefore a purely alternating one, H always being zero at the points given by

$$x = (2n+1)\frac{l}{2} + \frac{\alpha_1 - \alpha_2}{\pi}\cdot\frac{l}{2},$$

where n is an integer. The greatest value of H is obviously $2H_1$.

(4) *An alternating magnetic field whose maximum value is H is equivalent to two equal fields gliding in opposite directions whose maximum values are each $\frac{1}{2}H$.*

This follows from the theorem that

$$H\sin\omega t\cos\pi\frac{x}{l} = \frac{H}{2}\sin\left(\omega t + \frac{\pi x}{l}\right) + \frac{H}{2}\sin\left(\omega t - \frac{\pi x}{l}\right).$$

The angular velocities of the fields are $\frac{\omega}{p}$ and $-\frac{\omega}{p}$, where $2p$ is the number of poles.

(5) *The resultant of any number of alternating and gliding fields of the same period is in general two fields gliding in opposite directions.*

This follows from (1), (2) and (4). It is to be noted however that we make the assumption that they are all sine shaped.

If the two systems of currents represented by

$$i_1 = I_1 \sin\left(\omega t - \pi\frac{x}{l} + \alpha_1\right),$$

and

$$i_2 = I_2 \sin\left(\omega t + \pi\frac{x}{l} + \alpha_2\right),$$

are superposed in the distributed conductors, then the heat generated in the conductors will be that due to each system of currents separately, for the mean value of $(i_1 + i_2)^2$ is $\frac{I_1^2}{2} + \frac{I_2^2}{2}$. We shall call i_1 a system of currents turning to the right.

(6) *The mean torque produced by a magnetic field turning to the left on a system of currents turning to the right is zero.*

The currents may be flowing, for example, in the windings of the rotor of an induction motor.

The torque on any conductor will be proportional to the product

$$\sin\left(\omega t - \pi\frac{x}{l} + \alpha_1\right) \sin\left(\omega t + \pi\frac{x}{l} + \alpha_2\right).$$

The mean value of this expression from t equal to 0 to t equal to T is

$$\tfrac{1}{2}\cos\left(2\pi\frac{x}{l} + \alpha_2 - \alpha_1\right) \text{ ....(1).}$$

And the mean value of (1) from x equal to 0 to x equal to $2pl$ is obviously zero, and thus the mean torque between the field and the system of currents is zero. This theorem is analogous to the theorems in the undulatory theory of optics regarding the non-interference of circular vibrations in opposite directions. Several other theorems might be adapted from optics, and it will be found that these theorems are useful when we come to the theory of asynchronous motors. The artifice of replacing a fixed alternating field by two fields rotating in opposite directions, used first by Fresnel in optics and adapted by Ferraris to alternating current theory, is invaluable in this connection.

Rotating magnetic fields when the currents are not sine shaped.

Consider the case of two coils the axes of which intersect at some point O. We will consider the field at O in the plane determined by the two axes. Let h_1, h_2 be the values at O of the magnetic forces along the axes Op and Oq respectively, and let the angle pOq equal α. Now if Op and Oq be equal to h_1 and h_2 and r be the middle point of pq, $2Or$ will be the resultant force, and it will be seen that, as h_1 and h_2 alter, the locus of r may be a complicated curve and the angular velocity of Or may vary in an erratic manner. Draw any line Oa in the plane pOq, and let the angle pOa equal θ. Then, if h be the resultant magnetic force resolved along this line,

$$h = h_1 \cos\theta + h_2 \cos(\alpha - \theta).$$

Therefore

$$h^2 = h_1^2 \cos^2\theta + h_2^2 \cos^2(\alpha - \theta) + 2h_1 h_2 \cos\theta \cos(\alpha - \theta).$$

Now if the currents in the coils be adjusted until the effective or root mean square value of h_1 equals the R.M.S. value of h_2, and if capital letters denote the R.M.S. values,

$$H^2 = H_1^2 \{\cos^2\theta + \cos^2(\alpha - \theta) + 2\cos\theta\cos(\alpha - \theta)\cos\phi\},$$

where ϕ is the phase difference between h_1 and h_2 (see Chap. VI).

Noting that

$$\cos^2\theta = \sin^2\alpha + \cos(\alpha + \theta)\cos(\alpha - \theta),$$

we can easily prove that the above equation may be written in the form

$$H^2 = H_1^2 \{(\cos\alpha + \cos\overline{\alpha - 2\theta})(\cos\alpha + \cos\phi) + \sin^2\alpha\} \quad \ldots(1).$$

Now if $\alpha = \pi - \phi,$

so that $\cos\alpha + \cos\phi = 0,$

then $H = H_1 \sin\alpha.$

Thus, if the phase difference between the magnetic forces h_1 and h_2, or between the currents in the coils when no iron is used, be supplementary to the angle between their axes, then the effective value of the resultant magnetic force resolved along any line in the plane determined by the axes of the coils is constant. If this relation does not hold, then it is easy to show from (1) that H is a maximum when θ is $\frac{\alpha}{2}$ or $\pi + \frac{\alpha}{2}$, and a minimum when θ is

$\frac{\pi}{2}+\frac{\alpha}{2}$ or $-\frac{\pi}{2}+\frac{\alpha}{2}$. If we plot out a curve showing the values of H for various values of θ, we get a curve which is very similar to an ellipse.

This theorem can also be extended to three phase theory.

Extension to three phase theory.

Suppose that we have three equal cylindrical coils arranged round a circle at 120 deg. apart, their axes all pointing to the centre of the circle. Let h_1, h_2 and h_3 be the strengths of the fields produced by them at the centre, when they are connected to the mains of a three phase system in star fashion, then (Chap. XI),

$$\dot{i}_1+\dot{i}_2+\dot{i}_3=0,$$

and hence if there be no iron in the coils and if they be equal and similar

$$h_1+h_2+h_3=0.$$

Therefore $$H_1^2=H_2^2+H_3^2+2H_2H_3\cos\phi_{2.3}.$$

But $$H_1=H_2=H_3.$$

Thus $$\cos\phi_{2.3}=-\tfrac{1}{2},$$

and hence $$\phi_{2.3}=120^\circ.$$

If h be the magnetic force resolved along a line drawn through the centre making an angle θ with h_1, then

$$h=h_1\cos\theta+h_2\cos\left(\theta-\frac{2\pi}{3}\right)+h_3\cos\left(\theta-\frac{4\pi}{3}\right).$$

Thus $$H^2=H_1^2\left\{\cos^2\theta+\cos^2\left(\theta-\frac{2\pi}{3}\right)+\cos^2\left(\theta-\frac{4\pi}{3}\right)\right.$$
$$-\cos\theta\cos\left(\theta-\frac{2\pi}{3}\right)-\cos\left(\theta-\frac{2\pi}{3}\right)\cos\left(\theta-\frac{4\pi}{3}\right)$$
$$\left.-\cos\left(\theta-\frac{4\pi}{3}\right)\cos\theta\right\},$$

and hence $$H=\tfrac{3}{2}H_1.$$

Therefore H is independent of θ and is the same for every line in the plane of the circle.

In practical work, when the coils have iron cores, $h_1+h_2+h_3$ is not zero at every instant. We can, however, represent H_1, H_2 and

H_3 by lines forming a solid angle (Chap. VIII) and if the coils be equal and symmetrical and the currents magnetising them be equal, then, from symmetry, the phase differences between any two will be equal and will be less than 120 degrees. Hence we may put

$$\cos \phi_{2.3} = -\frac{m}{2},$$

where m is less than 1, and thus

$$H^2 = H_1^2 \left[\cos^2 \theta + \ldots - m \left\{\cos \theta \cos \left(\theta - \frac{2\pi}{3}\right) + \ldots \right\}\right]$$

$$= H_1^2 \left\{\frac{3}{2} + \frac{3m}{4}\right\},$$

and hence $H = \frac{1}{2} H_1 \sqrt{6 + 3m}.$

Therefore, in this case also, H is independent of θ.

Rotating magnetic fields producing a constant effective E.M.F. in a search coil placed with its plane perpendicular to the field.

If h be the strength of the field perpendicular to the plane of the coil, the E.M.F. induced in it will obviously be proportional to $\frac{dh}{dt}$. When we have two coils, we find in the same manner as before that

$$E^2 = E_1^2 \{(\cos \alpha + \cos \overline{\alpha - 2\theta})$$
$$(\cos \alpha + \cos \psi) + \sin^2 \alpha\} \ldots (2),$$

where the effective values of $\frac{dh_1}{dt}$ and $\frac{dh_2}{dt}$ are each equal to E_1, and ψ is the angle of phase difference between them. Hence when $\psi + \alpha$ equals π, the search coil indicates the same effective E.M.F. $E_1 \sin \alpha$, no matter what the angle θ may be, and since α can easily be measured ψ can be found. Now in a choking coil the applied potential difference e is proportional to $\frac{dh}{dt}$. If, therefore, we have two choking coils and we adjust them until the search coil connected to the terminals of an electrostatic voltmeter indicates the same effective E.M.F. for all positions of the search coil such that its plane is perpendicular to the plane of the axes of the choking coils, then the phase difference between the applied P.D.'s is the supplement of the angle between the axes.

Similarly, if three choking coils with their axes at angles of 120 degrees apart be magnetised by a system of three phase currents, we can show that the effective E.M.F. E induced in the search coil, no matter what the value of θ may be, is equal to $\frac{1}{2}E_1\sqrt{6+3m}$, where m is less than unity and E_1 is the E.M.F. that would be induced in the search coil by one of the coils acting alone when the plane of the search coil is perpendicular to the axis of the other coil. Thus it is possible, since E and E_1 can be found, to find m, and thus the phase difference between the rates at which the magnetisations in the coils are altering. This is equal to the phase difference between the electromotive forces.

Arnò's phase indicator.

In the form of phase indicator invented by Riccardo Arnò, two circular coils have a common diameter, about which they can rotate, and the currents, whose phase differences are to be measured, are passed through them. A search coil at the centre of their common axis and entirely enclosed by the two coils can also rotate about this axis which is in its plane. The search coil is connected to an electro-dynamometer or a hot wire ammeter, of negligible resistance, which measures the current in it. Now, in general, for different positions of the search coil, we get different readings in the ammeter. If, however, the current in one of the given coils be varied until the effective values of the magnetic forces produced at the common centre of the two coils be equal, it is possible, by adjusting the angle α between the planes of the two coils carrying the currents, to find a position in which the reading on the ammeter is always the same whatever the position of the search coil. If f be the resolved magnetic force perpendicular to the plane of the search coil, then

$$k\frac{df}{dt} = Ri + L\frac{di}{dt},$$

where R is the resistance and L the inductance of the search coil circuit and k is a constant. If R be negligible, then

$$k\frac{df}{dt} = L\frac{di}{dt}.$$

Therefore, integrating, $\qquad kf = Li + \text{const.},$

and, since f and i are alternating functions, the constant must be zero; thus taking effective values

$$kF = LA.$$

Now, since A is constant in all positions of the search coil, F must also be constant, and therefore, by the converse of the theorem proved on p. 299

$$\phi = 180° - \alpha,$$

where ϕ is the angle of phase difference between f_1 and f_2, *i.e.* the angle of phase difference between the currents, since there is no iron in the circuit, and α is the angle between the planes of the two coils.

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CHAPTER XV.

The magnetic fields round polyphase cables. The magnetic field round two long parallel wires carrying equal currents flowing in opposite directions. Bipolar circles. Currents equal and flowing in the same direction. Cassinian ovals. Lines of force when the currents are unequal. How the magnetic field alters round wires carrying single phase currents. Lines of force round three phase cables. Two phase cables. Twin concentric cables. Field of force round n parallel wires symmetrically arranged with their axes on a circle. Concentric cable. The strength of the magnetic field round n parallel wires. The strength of the field round a concentric main. The losses in cables. Three core cables. Duality. References.

The magnetic fields round polyphase cables.

THE magnetic fields round polyphase cables when carrying currents are very complex, but the equations to the lines of force can easily be found and it will be instructive to study them. In practice, we have to investigate whether the field will affect neighbouring telephone or telegraph wires, and whether it will produce appreciable eddy current losses in the lead sheath or in the copper shield, which is sometimes placed immediately inside the lead sheath to insure that in the event of one of the copper conductors getting accidentally into contact with it, the main fuse may act promptly. If there were no earth shield and one of the conductors made contact with the sheath, then, if the resistance of the earth in the neighbourhood of the place of contact were high, the sheath might be maintained at a high potential and be dangerous. In armoured cables we have also to investigate the hysteresis and eddy current losses in the steel strip or galvanised iron wires used to protect the cables.

We will first of all consider the magnetic field round two parallel wires. This would be the case of the mains of a two wire direct current system or of a single phase alternating current system. Let the wires be perpendicular to the plane of the paper, and let A and B (Fig. 113) be the points where their axes cut this plane, the current in A flowing towards the observer and in B away from him. We suppose that the wires are circular in section. So far therefore as the magnetic force at points external to them is concerned we may suppose that the currents are concentrated along the axes of the wires (p. 33). We shall only consider the magnetic forces at points external to the conductors. The magnetic force at any point P will be the resultant of two forces $\frac{2i}{AP}$ and $\frac{2i}{BP}$ which are

The magnetic field round two long parallel wires carrying equal currents flowing in opposite directions.

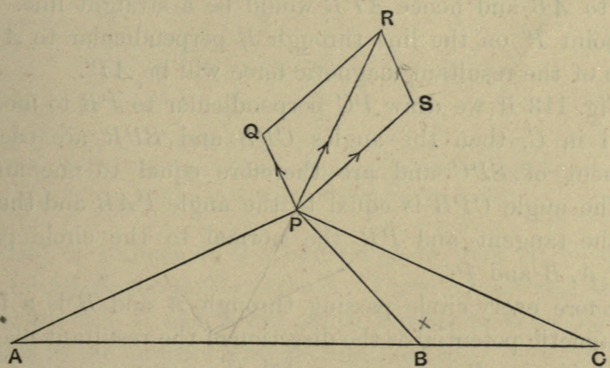

Fig. 113. Currents in opposite directions.

$$F = PR = \frac{2.AB.i}{r_1 r_2}.$$

perpendicular to AP and BP respectively and in the plane of the paper. Let PQ and PS represent these forces.

Let $AP = r_1$, $BP = r_2$, $AB = 2a$, $i =$ the value of the current in each wire in C.G.S. measure, and let $F = PR =$ the resultant magnetic force at P. Now the angles APB and PSR are each the supplement of the angle QPS and are therefore equal to one another.

Also
$$\frac{PS}{RS}=\frac{\frac{2i}{r_2}}{\frac{2i}{r_1}}=\frac{r_1}{r_2}=\frac{AP}{BP}.$$

Hence by Euclid (VI. 6) the triangles RPS and BAP are similar. Therefore the angle SPR equals the angle PAB and the angle QPR equals the angle PBA.

Again,
$$\frac{PR}{PS}=\frac{AB}{AP}$$
$$=\frac{2a}{r_1},$$

and therefore
$$F=\frac{4ai}{r_1 r_2}.$$

If the angle ABP were a right angle, then PS would be parallel to AB and hence APR would be a straight line. Thus at any point P' on the line through B perpendicular to AB, the direction of the resultant magnetic force will be AP'.

In Fig. 113 if we draw PC perpendicular to PR to meet AB produced in C, then the angles CPB and SPR are each the complement of SPC and are therefore equal to one another. Hence the angle CPB is equal to the angle PAB and therefore PC is the tangent and PR the normal to the circle passing through A, B and P.

Therefore every circle passing through A and B is a line of equal magnetic potential as the direction of the resultant magnetic force at any point P on it is normal to the curve.

Also, since the angle PCA is common to the triangles CPB, CAP, they are similar triangles, and thus

$$CP^2=CA\,.\,CB.$$

If, then, with centre C and radius $\sqrt{CA\,.\,CB}$ we describe a circle any radius CP of this circle will be a tangent to the circle through A, B and P; the tangent therefore at every point on the circle, whose centre is C, will be in the direction of the resultant magnetic force at that point. Hence this circle will be a line of force. The points A and B are inverse points with respect to the circle.

It follows that all circles which have A and B for inverse points (Fig. 114) will be lines of force round the wires. The polar of A with regard to all the circles surrounding B will pass through B and *vice versâ*. In other words

Bipolar circles.

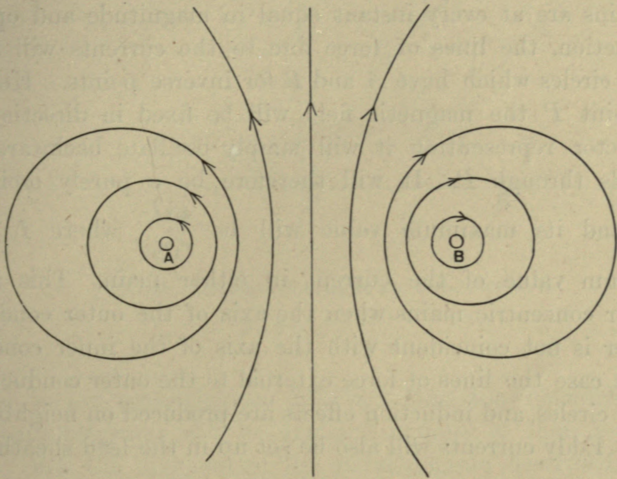

Fig. 114. The lines of force round wires carrying equal currents flowing in opposite directions are circles.

the chord of contact of the two tangents from A, to any circle surrounding B, passes through B. These circles and the circles passing through A and B are called bipolar circles. The bipolar equation to the circles round B (Fig. 113) is

$$\frac{r_1}{r_2} = \frac{AP}{BP} = \frac{CA}{CP} = \text{constant} = m,$$

or

$$r_1 = mr_2.$$

The equation to any line of equal magnetic potential will obviously be

$$\theta_2 - \theta_1 = \alpha,$$

where α is a constant and θ_2, θ_1 are the angles PBC, PAC respectively.

We have shown above that the magnitude of the resultant magnetic force is given by $\frac{4ai}{r_1 r_2}$.

Thus the bipolar equation of any line of equal magnetic force is

$$r_1 r_2 = \text{constant}.$$

These curves are Cassinian ovals and are shown in Fig. 116.

Since with single phase alternating currents the currents in the mains are at every instant equal in magnitude and opposite in direction, the lines of force due to the currents will always be the circles which have A and B for inverse points. Hence at any point P the magnetic field will be fixed in direction, and the vector representing it will simply oscillate backwards and forwards through P. It will therefore be a purely oscillatory field, and its maximum value will be $\frac{4aI}{r_1 r_2}$, where I is the maximum value of the current in either main. This is also true for concentric mains when the axis of the outer conducting cylinder is not coincident with the axis of the inner conductor. In this case the lines of force external to the outer conductor are bipolar circles, and induction effects are produced on neighbouring wires. Eddy currents will also be set up in the lead sheath.

Currents equal and flowing in the same direction.

Let F be the resultant magnetic force at P (Fig. 115) and let the angle APB equal α, then, with the same notation as before,

$$F^2 = \frac{4i^2}{r_1^2} + \frac{4i^2}{r_2^2} + 2\frac{4i^2}{r_1 r_2}\cos\alpha$$

$$= \frac{4i^2}{r_1^2} + \frac{4i^2}{r_2^2} + \frac{4i^2}{r_1^2 r_2^2}(r_1^2 + r_2^2 - 4a^2)$$

$$= \frac{8i^2}{r_1^2 r_2^2}(r_1^2 + r_2^2 - 2a^2)$$

$$= \frac{8i^2}{r_1^2 r_2^2} \cdot 2 \cdot OP^2.$$

Therefore $$F = \frac{4i \cdot OP}{r_1 r_2}.$$

Also, since in Fig. 115

$$\frac{AP}{PB} = \frac{PS}{PQ},$$

and the angle APB equals the angle QPS, it follows that the parallelogram described on AP and PB as adjacent sides will be

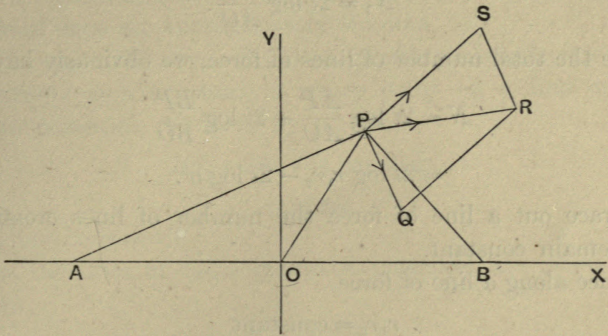

Fig. 115. Currents flowing in the same direction.

$$F = PR = \frac{4 . OP . i}{r_1 r_2}.$$

similar to the parallelogram $PQRS$. Thus the diagonals of the two parallelograms will be inclined equally to their respective sides. But the diagonal of the parallelogram described on AP and PB as adjacent sides is represented by $2 . PO$ in magnitude and direction. Therefore

$$S\hat{P}R = O\hat{P}A \text{ and } R\hat{P}Q = O\hat{P}B.$$

Also $$O\hat{P}A = \frac{\pi}{2} - B\hat{P}R \text{ and } O\hat{P}B = A\hat{P}R - \frac{\pi}{2}.$$

Hence, for example, in the particular case when APR is a straight line, OPB is a right angle, and therefore at every point P' on the circle described on OB as diameter, AP' gives the direction of the resultant force.

Cassinian ovals.

The bipolar equation to the lines of force can be found as follows.

If N_1 be the number of lines of force, per unit length of the wires, crossing OP (Fig. 115) due to the current i in the wire whose axis cuts the plane of the paper perpendicularly at A, we have

$$N_1 = \int \frac{2i}{r}\, dr = [2i \log r],$$

where A is the origin and the limits of r are AO and AP. Thus we get

$$N_1 = 2i \log \frac{AP}{AO}.$$

If N be the total number of lines of force, we obviously have

$$N = 2i \log \frac{AP}{AO} + 2i \log \frac{BP}{BO}$$
$$= 2i \log r_1 r_2 - 2i \log a^2.$$

If P trace out a line of force the number of lines crossing CP must remain constant.

Hence along a line of force

$$r_1 r_2 = \text{constant}$$
$$= m^2,$$

and this is the required equation.

If we transform this equation into polar coordinates with O as pole and OB as initial line (Fig. 115), we get

$$r^4 - 2a^2 r^2 \cos 2\theta + a^4 - m^4 = 0.$$

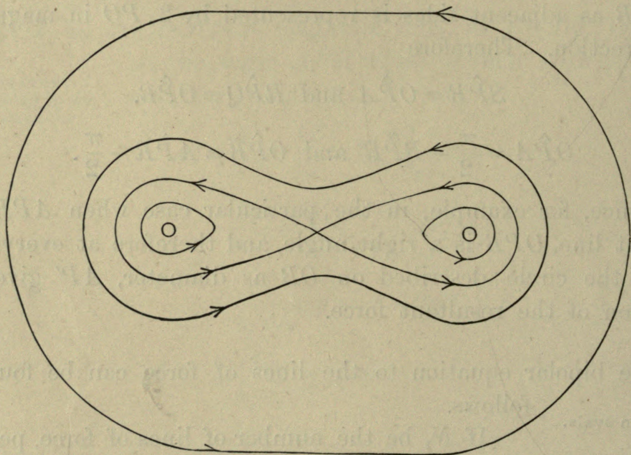

Fig. 116. The lines of force when equal currents are flowing in the same direction are Cassinian ovals.

If m be less than a, we get two ovals (Fig. 116), when m equals a we get the lemniscate, the lines of force apparently crossing at the origin where the magnetic force is zero, and when m is greater than a we get only one oval. If m be greater than $\sqrt{2}a$ there is no point of inflexion on the oval, and if m be very large the lines of force are approximately circular.

Again, since the sum of the magnetic potentials due to the two currents is a constant at every point on a line of equal magnetic potential, we have by p. 32

$$\begin{aligned} 2i\theta_2 + 2i\theta_1 &= \text{a constant} \\ &= 2i\alpha, \end{aligned}$$

where α is a constant.

Thus the lines of equal magnetic potential are given by the equation

$$\theta_2 + \theta_1 = \alpha,$$

which represents a series of equiaxial hyperbolas passing through A and B and cutting the Cassinian ovals at right angles.

Lines of force when the currents are unequal.

When the currents in the long parallel wires are unequal in magnitude, the equations to the lines of force can be written down at once in bipolar coordinates by the above method. We shall first, however, use Cartesian coordinates as this method is instructive and will be found convenient in solving problems when the earth's field has to be taken into account. Let us suppose that the instantaneous value of the current in the wire A (Fig. 115) is i_1 and in the wire B, i_2, then if (x, y) be the coordinates of the point P, we have by resolving the magnetic forces at P horizontally and vertically,

$$F\cos\theta = \frac{2i_1}{r_1}\sin\theta_1 + \frac{2i_2}{r_2}\sin\theta_2,$$

and
$$F\sin\theta = -\frac{2i_1}{r_1}\cos\theta_1 - \frac{2i_2}{r_2}\cos\theta_2,$$

where θ is the angle which F makes with OX, θ_1 is the angle PAB and θ_2 is the angle PBX.

Also

$$r_1^2 = y^2 + (a+x)^2; \quad \sin\theta_1 = \frac{y}{r_1}; \quad \cos\theta_1 = \frac{a+x}{r_1},$$

and
$$r_2^2 = y^2 + (a-x)^2; \quad \sin\theta_2 = \frac{y}{r_2}; \quad \cos\theta_2 = -\frac{a-x}{r_2}.$$

Hence if (x, y) be a point on the line of force through P,

$$\frac{dy}{dx} = \tan\theta$$

$$= \frac{-\dot{i}_1 r_2^2 (a+x) + \dot{i}_2 r_1^2 (a-x)}{\dot{i}_1 r_2^2 y + \dot{i}_2 r_1^2 y}$$

$$= -\frac{x}{y} + \frac{a}{y}\,\frac{l(x^2+y^2+a^2)+2ax}{x^2+y^2+a^2+2lax} \quad \ldots\ldots\ldots(a),$$

where $$l = \frac{\dot{i}_2 - \dot{i}_1}{\dot{i}_2 + \dot{i}_1},$$

and thus $$\frac{d}{dx}(ux) = \frac{lu+1}{u+l},$$

where $$u = \frac{x^2+y^2+a^2}{2ax}.$$

Hence $$x\frac{du}{dx} = -\frac{u^2-1}{u+l},$$

and therefore $$\frac{dx}{x} = -\frac{u+l}{u^2-1}\,du.$$

Hence integrating and simplifying we get

$$(ux - x)^{l+1} = m^2 (ux + x)^{l-1},$$

where m is a constant. Substituting for u and l their values we get finally

$$r_1^{\dot{i}_1} r_2^{\dot{i}_2} = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

as the equation which gives all the lines of force.

From the equation (a) given above we see that the tangents to the lines of force at points where they cut the curve

$$x^2 + y^2 + a^2 + 2lax = 0,$$

or $$\left(x - a\frac{\dot{i}_1 - \dot{i}_2}{\dot{i}_1 + \dot{i}_2}\right)^2 + y^2 = -a^2\frac{4\dot{i}_1\dot{i}_2}{(\dot{i}_1+\dot{i}_2)^2},$$

are at right angles to the line joining the axes of the two wires. If the currents are flowing in the same direction, the curve is imaginary, but if they are flowing in opposite directions it is a circle with its centre at the point

$$a\frac{\dot{i}_1 - \dot{i}_2}{\dot{i}_1 + \dot{i}_2}, \quad 0,$$

and the radius of this circle equals $2a \frac{(-i_1 i_2)^{\frac{1}{2}}}{i_1 + i_2}$. If C be its centre

$$AC = a \frac{2i_1}{i_1 + i_2} \text{ and } BC = a \frac{-2i_2}{i_1 + i_2},$$

and therefore $$\frac{2i_1}{AC} = \frac{-2i_2}{BC}.$$

The centre of this circle is therefore the point where the magnetic forces due to the currents in the two wires balance.

In the differential equation (a) if we write on the left-hand side $-\frac{dx}{dy}$ instead of $\frac{dy}{dx}$ we get the differential equation of the lines of equal magnetic potential round the wires. The solution in this case may be written in the form

$$i_1\theta_1 + i_2\theta_2 = \text{constant} \quad \text{.....................(2)}.$$

The curves given by this equation are the same as the stream lines of current in an infinite metal plate, two points A and B of which are maintained at potentials which are proportional to i_1 and i_2 respectively.

The above equations (1) and (2) can also be proved as follows by using bipolar coordinates. Since there is no magnetic force at right angles to a line of force, we have

$$\frac{2i_1}{r_1}\cos\psi_1 + \frac{2i_2}{r_2}\cos\psi_2 = 0 \quad \text{..................(3)},$$

where ψ_1 and ψ_2 are the angles between the radii AP and BP and the resultant magnetic force PR (Fig. 115) which is a tangent to the required curve.

Now $$\cos\psi_1 = \frac{dr_1}{ds}, \text{ and } \cos\psi_2 = \frac{dr_2}{ds}.$$

Hence substituting we get

$$\frac{i_1}{r_1}\frac{dr_1}{ds} + \frac{i_2}{r_2}\frac{dr_2}{ds} = 0,$$

and therefore $$i_1 \log r_1 + i_2 \log r_2 = \text{constant}.$$

Thus $$r_1^{i_1} r_2^{i_2} = \text{constant}.$$

If ψ_1', ψ_2' be the angles which the normal to the line of force at P makes with r_1 and r_2, then $\psi_1' = \psi_1 - \frac{\pi}{2}$, and $\psi_2' = \psi_2 - \frac{\pi}{2}$, hence substituting in (3) we get

$$\frac{2i_1}{r_1}\sin\psi_1' + \frac{2i_2}{r_2}\sin\psi_2' = 0.$$

Now if ds' be an element of the equipotential curve through P,

$$\sin\psi_1' = r_1\frac{d\theta_1}{ds'}, \text{ and } \sin\psi_2' = r_2\frac{d\theta_2}{ds'},$$

and therefore $$i_1 d\theta_1 + i_2 d\theta_2 = 0.$$

Hence $$i_1\theta_1 + i_2\theta_2 = \text{constant}.$$

In Fig. 117 the lines of force are shown when the currents are flowing in opposite directions and i_1 is four times i_2.

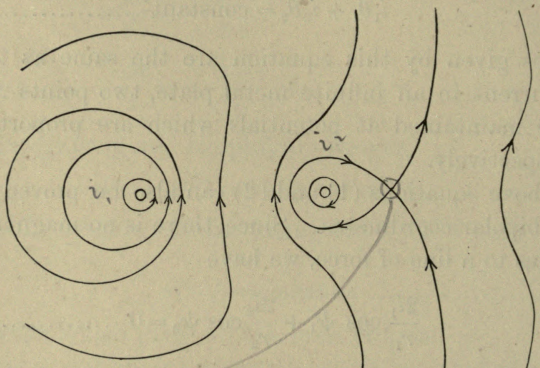

Fig. 117. Currents flowing in opposite directions, one being four times greater than the other.

The neutral point, that is, the point where the magnetic force is zero, ought to be specially noted. The bipolar equation to the looped line of force through this point is

$$27r_1^4 = 2048r_2a^3.$$

In general, if N be the neutral point and A and B the points where the axes of the wires cut the plane of the paper, then,

$$\frac{AN}{BN} = \frac{i_1}{i_2}.$$

If the currents are flowing in the same direction, N is between the wires, if in opposite directions N is on the side next the weaker current. When N is at an infinite distance, $i_1 = -i_2$ and the lines of force are the circles shown in Fig. 114. When $i_1 = -4i_2$ the lines of force are as in Fig. 117. When i_2 equals zero the lines of force are circles round A. When $i_1 = 4i_2$, N divides AB in the ratio of 4 to 1, and a crossed line of force through this point loops A with B. Finally, when $i_1 = i_2$ the lines of force are the Cassinian ovals shown in Fig. 116.

How the magnetic field alters when i_1 and i_2 are alternating currents.

If the ratio of i_1 to i_2 is always constant, that is, if the phase difference between i_1 and i_2 is either zero or 180 degrees, then the lines of force are fixed. In this case the field at any point is a purely oscillatory one.

If however the phase difference between the currents is neither zero nor 180 degrees, the magnetic field is continually changing in direction. It is not difficult to form a mental picture of what happens by considering how the neutral point N oscillates in any given case and by studying the Figures 114, 116 and 117. In general the force at any point is continually changing in magnitude and direction and hence is partly oscillatory and partly rotary.

Magnetic field round a three phase cable.

Suppose that the three copper conductors are parallel cylinders and that their axes cut the paper at the angular points A, B and C of an equilateral triangle whose sides are of length d. Then if i_1, i_2 and i_3 be the instantaneous values of the currents, the magnetic force (f) at any instant will be the resultant of the three magnetic forces $\frac{2i_1}{r_1}$, $\frac{2i_2}{r_2}$ and $\frac{2i_3}{r_3}$ acting at P and perpendicular to AP, BP and CP respectively. By a well-known theorem in Statics, we have

$$f^2 = \frac{4i_1^2}{r_1^2} + \ldots + 2\,\frac{4i_2 i_3}{r_2 r_3}\cos\theta_{2.3} + \ldots.$$

But
$$\cos\theta_{2.3} = \frac{r_2^2 + r_3^2 - d^2}{2r_2 r_3},$$

and thus
$$f^2 = \frac{4i_1^2}{r_1^2} + \ldots + 4i_2 i_3\left(\frac{1}{r_2^2} + \frac{1}{r_3^2} - \frac{d^2}{r_2^2 r_3^2}\right) + \ldots.$$

Now, in practice, we generally have

$$i_1 + i_2 + i_3 = 0,$$

and therefore $$2i_2 i_3 = i_1^2 - i_2^2 - i_3^2.$$

On substituting this value of $2i_2 i_3$ in the above equation for f^2 and simplifying, it is easy to see that

$$f^2 = 2i_1^2 d^2 \left(\frac{1}{r_1^2 r_2^2} + \frac{1}{r_1^2 r_3^2} - \frac{1}{r_2^2 r_3^2}\right) + \ldots + \ldots.$$

Let O be the centre of the circle through A, B and C, a its radius and let the angle POA be θ, then we have

$$r_1^2 = r^2 - 2ar\cos\theta + a^2,$$

$$r_2^2 = r^2 - 2ar\cos(\theta - \tfrac{2}{3}\pi) + a^2,$$

and $$r_3^2 = r^2 - 2ar\cos(\theta - \tfrac{4}{3}\pi) + a^2.$$

Substituting these values of r_1, r_2 and r_3 in the formula for f^2 and noting that d^2 is $3a^2$ and that

$$r_1^2 r_2^2 r_3^2 = r^6 - 2a^3 r^3 \cos 3\theta + a^6,$$

we get

$$f^2 = \frac{6a^2(a^2 + r^2)}{r^6 - 2a^3 r^3 \cos 3\theta + a^6}(i_1^2 + i_2^2 + i_3^2)$$

$$+ \frac{24a^3 r}{r^6 - 2a^3 r^3 \cos 3\theta + a^6}\{i_1^2 \cos\theta + i_2^2 \cos(\theta - \tfrac{2}{3}\pi) + i_3^2 \cos(\theta - \tfrac{4}{3}\pi)\}.$$

At the centre of the circle through A, B and C, r is zero and thus

$$f^2 = \frac{6(i_1^2 + i_2^2 + i_3^2)}{a^2}.$$

In order that the field at the centre of the circle may be of constant strength we must have

$$i_1^2 + i_2^2 + i_3^2 = \text{constant}.$$

This is true, for example, when the currents follow the harmonic law.

Let us now suppose that the currents follow the harmonic law so that we can write

$$i_1 = I\cos\omega t, \quad i_2 = I\cos(\omega t - \tfrac{2}{3}\pi), \quad i_3 = I\cos(\omega t - \tfrac{4}{3}\pi).$$

Then we have

$$i_1^2 + i_2^2 + i_3^2 = \tfrac{3}{2}I^2,$$

and $$i_1^2\cos\theta + \ldots + \ldots = \tfrac{1}{2}I^2\{(1 + \cos 2\omega t)\cos\theta + \ldots + \ldots\}$$

$$= \tfrac{3}{4}I^2\cos(2\omega t + \theta).$$

Substituting these values in the formula for f^2, we get

$$f^2 = \left(\frac{3I}{a}\right)^2 \cdot \frac{a^4\{a^2 + r^2 + 2ar\cos(2\omega t + \theta)\}}{r^6 - 2a^3r^3\cos 3\theta + a^6}.$$

At points on the circumference of the circle passing through ABC, r equals a, and thus

$$\pm f = \frac{3I}{a} \cdot \frac{\cos\left(\omega t + \frac{\theta}{2}\right)}{\sin\frac{3\theta}{2}}.$$

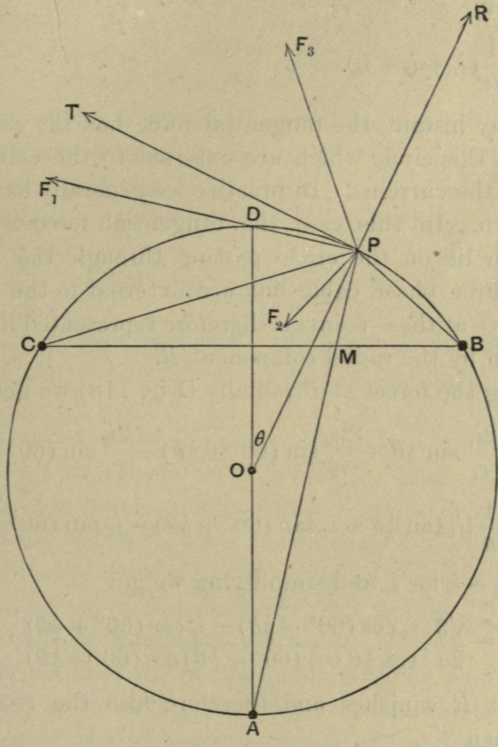

Fig. 118. The currents in the wires are supposed to be flowing in the same direction.

Let us now consider the magnetic force at a point P on the circumference of this circle when the currents in the cores are i_1, i_2 and i_3 respectively (Fig. 118).

Since ABC is an equilateral triangle the angle CPB is 120° and AP bisects it. Let O be the centre of the circle and let the angle POD be θ and let OA be a. The angle OPA is obviously $\frac{1}{2}\theta$ so that the angles CPO and BPO are $60° - \frac{1}{2}\theta$ and $60° + \frac{1}{2}\theta$. We have also $r_1 = AP = 2a \cos \frac{1}{2}\theta$, $r_2 = 2a \cos (60° + \frac{1}{2}\theta)$ and $r_3 = 2a \cos (60° - \frac{1}{2}\theta)$.

Let PT be the tangent at the point P and let OP be produced to R. The tangential force T along PT will be given by

$$T = \frac{2i_1}{r_1} \cos \tfrac{1}{2}\theta + \frac{2i_2}{r_2} \cos (60° + \tfrac{1}{2}\theta) + \frac{2i_3}{r_3} \cos (60° - \tfrac{1}{2}\theta)$$

$$= \frac{1}{a}(i_1 + i_2 + i_3).$$

Hence, at any instant, the tangential force has the same value at all points on this circle which are external to the cores, whatever the value of the currents. In practice we generally have $i_1 + i_2 + i_3$ equal to zero. In this case the tangential force is zero at all points which lie on the circle passing through the axes of the cores of a three phase cable but are external to the cores. The resultant force at these points is therefore represented in magnitude and direction by the radial component R.

Resolving the forces at P radially (Fig. 118) we get

$$-R = \frac{2i_1}{r_1} \sin \tfrac{1}{2}\theta + \frac{2i_2}{r_2} \sin (60° + \tfrac{1}{2}\theta) - \frac{2i_3}{r_3} \sin (60° - \tfrac{1}{2}\theta)$$

$$= \frac{1}{a}\{i_1 \tan \tfrac{1}{2}\theta + i_2 \tan (60° + \tfrac{1}{2}\theta) - i_3 \tan (60° - \tfrac{1}{2}\theta)\}.$$

Writing $-i_2 - i_3$ for i_1 and simplifying we get

$$-R = \frac{\sqrt{3}}{2a} \cdot \frac{i_2 \cos (60° - \frac{1}{2}\theta) - i_3 \cos (60° + \frac{1}{2}\theta)}{\cos \frac{1}{2}\theta \cos (60° - \frac{1}{2}\theta) \cos (60° + \frac{1}{2}\theta)}.$$

We see that R vanishes and therefore also the resultant force vanishes when

$$i_2 \cos (60° - \tfrac{1}{2}\theta) - i_3 \cos (60° + \tfrac{1}{2}\theta) = 0,$$

and therefore when $$\frac{i_2}{i_3} = \frac{\cos (60° + \frac{1}{2}\theta)}{\cos (60° - \frac{1}{2}\theta)}$$

$$= \frac{r_2}{r_3} = \frac{BM}{CM}.$$

This theorem gives us a simple construction for finding the neutral point. Join the axes B and C of the two cores in which the currents are flowing in the same directions. Divide this line at M so that BM is to CM in the ratio of i_2 to i_3. Produce AM to meet the circle at P; then, since PM bisects the angle CPB, P is the neutral point.

If we write $I \cos(\omega t - \frac{2}{3}\pi)$ and $I \cos(\omega t - \frac{4}{3}\pi)$ for i_2 and i_3 in the above formula, we get

$$-R = \frac{3I}{a} \cdot \frac{\sin(\omega t - \frac{1}{2}\theta)}{\cos \frac{1}{2}\theta \, (2 \cos \theta - 1)}$$

$$= \frac{3I}{a} \cdot \frac{\sin(\omega t - \frac{1}{2}\theta)}{\cos \frac{3}{2}\theta}.$$

Writing $180° - \theta$ for θ in this formula, we see, by comparing it with the formula we have found for $\pm f$ (p. 317) that R must be equal to f. This also follows since, in this case, T is zero.

We see that the neutral point is determined by

$$\theta = 2\omega t, \text{ so that } \frac{d\theta}{dt} = 2\omega,$$

and thus as long as it is outside the cores of the conductors it moves round the circumference of the circle with an angular velocity 2ω.

The amplitude of the magnetic force at a point P on the circumference of the circle is $3I/a \cos \frac{3}{2}\theta$. Thus the magnetic force has the minimum amplitude $3I/a$ where θ is zero. The amplitude of the magnetic force increases as θ increases. When θ is 50° it is nearly four times as large as the minimum value and for higher values of θ it increases very rapidly. The formula is not applicable to points inside the cores.

Let us now consider the magnetic force at any point P, not on the circle ABC, when the currents follow the harmonic law. This force is the resultant of three forces fixed in direction which oscillate according to the same law. The extremity of the line representing the resultant force at P therefore (Chapter XIV) traces out an ellipse, and if f be the value of the resultant force and ω_1 its angular velocity at any instant we have

$$f^2 \omega_1 = \text{constant}.$$

This is true even for points inside the metal cores. It follows that if f vanish at any instant it must either be zero always or it must oscillate in a straight line. Since there is no neutral point outside two of the cores when the current in the third vanishes, it follows that there is no point outside the cores where the magnetic force is always zero. Similarly there is no point inside the cores where the magnetic force is always zero. If the force, therefore, ever vanish at a point the field at that point must be purely oscillatory.

Along the axis of a three phase cable carrying currents which follow the sine law, the field is purely rotary, that is, the extremity of the line representing the resultant force describes a circle. As we approach the circle ABC the fields become elliptical and their eccentricity increases as we approach the circumference. At points on this circle, outside the cores, the magnetic forces are purely oscillatory, the oscillations taking place in the radial direction. Outside this circle the field at any point is elliptical but the eccentricity is practically zero when the distance r of the point from the cable is large compared with the radius a of the circle ABC. We shall see later on that, when a^2/r^2 can be neglected compared with a/r, we get a rotating field the strength of which is $3aI/r^2$ where I is the maximum value of the current in a core.

The locus of the positions of the neutral points *inside* the cores lies outside of the circle ABC. At all points on this curve the field is purely oscillatory.

The equations to the lines of force can easily be found in terms of tripolar coordinates. Since the resultant magnetic force at any point of a line of force is by definition a tangent to the curve, it follows that the sum of the component forces resolved at right angles to a line of force vanishes. At every point on a line of force we therefore have

$$f_1 \cos \psi_1 + f_2 \cos \psi_2 + f_3 \cos \psi_3 = 0,$$

where ψ_1 is the angle between AP and the tangent to the line of force through P, etc., and thus we get

$$\frac{2i_1}{r_1} \cdot \frac{dr_1}{ds} + \frac{2i_2}{r_2} \cdot \frac{dr_2}{ds} + \frac{2i_3}{r_3} \cdot \frac{dr_3}{ds} = 0.$$

Therefore $$i_1 \log r_1 + i_2 \log r_2 + i_3 \log r_3 = \text{constant},$$
or $$r_1^{i_1} r_2^{i_2} r_3^{i_3} = \text{constant}.$$
For given values of i_1, i_2 and i_3 we get the equations to all the lines of force by giving different values to the constant.

Since, in practice,
$$i_1 + i_2 + i_3 = 0,$$
the equation to the lines of force may be written
$$r_1^{i_1} r_2^{i_2} = m r_3^{i_1+i_2}.$$
To illustrate the curves represented by this equation, we shall draw the lines of force in particular cases. Suppose, for example, that
$$i_1 = i_2 = -\tfrac{1}{2} i_3,$$
then, at this instant, all the lines of force are given by
$$r_1 r_2 = m r_3^2.$$

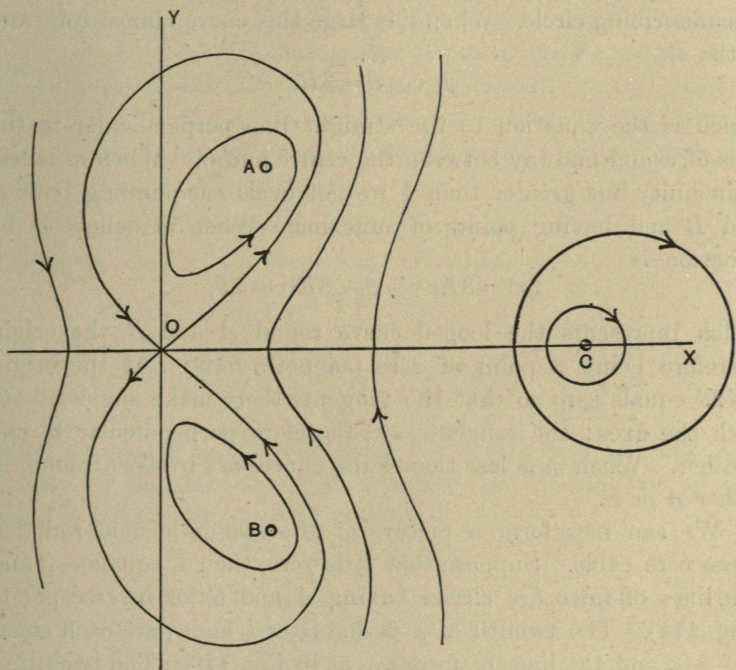

Fig. 119. Lines of force in a three phase cable when $i_1 = i_2 = -\tfrac{1}{2} i_3$.

If we take as origin (Fig. 119) a point O on the circle circumscribing A, B and C, so that O is equidistant from A and B, and if we take OC as the axis of x, the equation to the lines of force is

$$\left\{(x-\tfrac{1}{2}R)^2+\left(y-\frac{\sqrt{3}}{2}R\right)^2\right\}\left\{(x-\tfrac{1}{2}R)^2+\left(y+\frac{\sqrt{3}}{2}R\right)^2\right\}$$
$$=m^2\{(x-2R)^2+y^2\}^2,$$

where R is the radius of the circumscribing circle. Putting this equation into polar coordinates we get

$$(r^2-rR\cos\theta+R^2)^2-m^2(r^2-4rR\cos\theta+4R^2)^2=3r^2R^2\sin^2\theta.$$

Some of these curves are shown in Fig. 119. When m is greater than unity the curves are ovals round C. When m equals unity we get

$$(2r^2-5Rr\cos\theta+5R^2)(r\cos\theta-R)=Rr^2\sin^2\theta$$

as the equation of the line of force through the centre of the circumscribing circle. When r is large this curve almost coincides with

$$2r\cos\theta=3R,$$

which is the equation to the straight line perpendicular to the axis of x and midway between the centre and C. When m is less than unity but greater than $\frac{1}{4}$ we get ovals surrounding both A and B and having points of inflexion. When m equals $\frac{1}{4}$ the equation is

$$5r^2=8Rr\cos\theta-8R^2\cos 2\theta,$$

which represents the looped curve round A and B, the origin therefore being a point of zero magnetic force. At the origin $\cos 2\theta$ equals zero so that the tangents there make angles of 45° with the axes; the tangents are therefore perpendicular to one another. When m is less than $\frac{1}{4}$ the curves are ovals surrounding either A or B.

We can now form a picture of the magnetic field round a three core cable. Suppose that i_2 is zero, then i_1 equals $-i_3$ and the lines of force are circles having A and C for inverse points (Fig. 114). The twelfth of a period later i_2 and i_1 are each equal to $-\frac{1}{2}i_3$, and the lines of force are as in Fig. 119. The twelfth of a period after this, they will be circles round B and C as inverse points and so on. The neutral point, outside the cores, makes two complete revolutions round the circle passing through A, B and C during the period of the alternating current.

It must be remembered that in practice the cores, instead of being cylinders, are stranded cables made up of $1+6+12+18+\dots$ wires, the usual numbers being nineteen and thirty-seven. The outside layers generally have a 'lay' of about twenty times the diameter, that is, the wires make a complete twist round the axis of a core in a length equal to twenty times the core's diameter. In addition, the three stranded conductors or cores are spiralled relatively to one another, each making a complete turn round the axis of the cable in about eight feet. The section of a conductor is sometimes shaped like the sector of a circle, the vertical angle of the sector being 120 degrees (Fig. 33). The main features of the field of force round the cable are, however, similar to the fields we have described above.

Magnetic field round a two phase cable.

In this cable (see Fig. 35) we have four cores and the sections of the axes by the plane of the paper are at the angular points of a square. If r_1, r_2, r_3 and r_4 be the distances of these four points from P and if i_1, i_2, i_3 and i_4 be the values of the currents at any instant, the lines of force will be given by

$$r_1^{i_1} r_2^{i_2} r_3^{i_3} r_4^{i_4} = \text{constant}.$$

During the normal working of the system, we have

$$i_1 + i_3 = 0 \text{ and } i_2 + i_4 = 0,$$

and thus the equation becomes

$$\left(\frac{r_1}{r_3}\right)^{i_1} \left(\frac{r_2}{r_4}\right)^{i_2} = \text{constant}.$$

When i_2 is zero the lines of force are circles having 1 and 3 as inverse points. An eighth of a period later all the currents are equal in magnitude and therefore

$$r_1 r_2 = m r_3 r_4$$

gives the equations to the lines of force. When m equals unity we get a straight line passing through the centre and dividing the field into two symmetrical portions. There are in general two neutral points on the circle circumscribing 1, 2, 3 and 4, and the resultant magnetic force at all other points on this circle, which are external to the cores, is normal to the circle. When i_1 equals i_2, the equation to the looped lines of force passing through the neutral points which are external to the cores is .

$$r_1 r_2 = (3 \pm 2\sqrt{2})\, r_3 r_4.$$

Hence it is easy to draw a rough diagram of the lines of force in this case. An eighth of a period later, the lines of force are circles having 2 and 4 as inverse points, and so on.

When the currents follow the harmonic law, the neutral points, when outside the cores, travel with uniform speed round the circle which passes through the axes of the four cores. At the instants when i_1 and i_3 or i_2 and i_4 vanish, the lines of force are circles (Fig. 114) and there are neutral points only inside the cores.

Twin concentric cables.

In the case of a twin concentric cable (Fig. 37), the phase difference between the currents in the two inner conductors is ninety degrees, and the current in the outer cylindrical conductor enveloping them is equal, at every instant, to the sum of the two inner currents. The current in the outer return conductor produces no magnetic field inside its inner radius. The magnetic field inside the outer conductor will therefore be due to the currents in the two inner conductors. The neutral point, due to these two currents, will oscillate on the line joining their axes and it is easy to draw the lines of force inside at any instant.

If O be the centre of the cable and A and B the points where the axes of the inner conductors cut the plane of the paper, O will be the middle point of AB. Let $OA = a$ and $ON = x$, where N is the neutral point outside the cable, then

$$\frac{2\dot{i}_1}{x+a} + \frac{2\dot{i}_2}{x-a} = \frac{2(\dot{i}_1 + \dot{i}_2)}{x}.$$

Therefore
$$x = a\,\frac{\dot{i}_1 + \dot{i}_2}{\dot{i}_1 - \dot{i}_2}.$$

Hence the position of the neutral point is determined and the field outside the cable can be drawn.

When $\dot{i}_2$ equals $\dot{i}_1$, the equation to the lines of force outside the cable is

$$r_1 r_2 = mOP^2,$$

or

$$(r^2 + a^2)^2 - 4a^2 r^2 \cos^2\theta = m^2 r^4.$$

When m equals unity this gives an equiaxial hyperbola.

The equation to the lines of force in a plane perpendicular to the axes of the wires, which we may suppose to be the plane of the paper, is

Field of force round n parallel wires symmetrically arranged with their axes on a circle.

$$r_1^{i_1} r_2^{i_2} \dots r_n^{i_n} = \text{constant},$$

where r_1, $r_2 \dots r_n$ are the distances of a point P in this plane from the axes of the wires, and i_1, $i_2 \dots i_n$ are the currents in the wires. If the currents are all equal and flowing in the same direction this equation becomes

$$r_1 r_2 \dots r_n = \text{constant}.$$

By De Moivre's property of the circle, we can write

$$\{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}\}^{\frac{1}{2}} = r_1 r_2 \dots r_n$$
$$= \text{constant},$$

where a is the radius of the circle, r the distance of P from its centre O, and θ the angle between OP and a line joining O to one of the points of intersection of an axis of the conductor with the plane of the paper.

At the centre of the circle,

$$r_1 = r_2 = \dots = r_n = a.$$

The equation to the lines of force through the centre is therefore

$$r^n = 2a^n \cos n\theta.$$

This gives us n loops having a multiple point at the centre, where the force is zero.

The equation to the line of force passing through the point

$$\left(r = a, \quad \theta = \frac{\pi}{n}\right)$$

is
$$r^{2n} - 2a^n r^n \cos n\theta + a^{2n} = 4a^{2n}.$$

This is a curve with n ripples, the minimum value of r being a and the maximum value being $3^{\frac{1}{n}} a$. If n were 40, the maximum deviation of this line of force from the circle passing through the axes of the wires would be only 2·8 per cent. of the radius of that circle.

The equation to the line of force passing through the point

$$\left(r = 1{\cdot}5a, \quad \theta = \frac{\pi}{n}\right)$$

is
$$r^{2n} - 2a^n r^n \cos n\theta + a^{2n} = \{(1{\cdot}5)^n + 1\}^2 a^{2n}.$$

It is easy to see that $1{\cdot}5a$ is the minimum value of r and that $\{(1{\cdot}5)^n + 2\}^{\frac{1}{n}} a$ is its maximum value. When n is 8 the maximum value of r is $1{\cdot}514a$, and hence this line of force never deviates from the circle whose radius is $1{\cdot}507a$ by as much as the half of one per cent. of that radius.

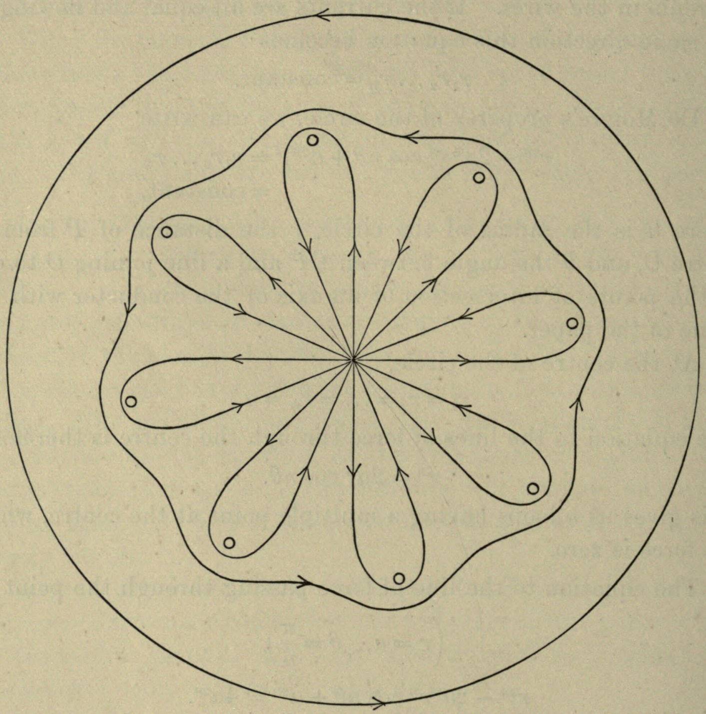

Fig. 120. Lines of force round eight wires carrying currents flowing in the same direction.

The lines of force when there are eight wires are shown in Fig. 120, the equation to the curves being

$$r^{16} - 2a^8 r^8 \cos 8\theta + a^{16} = m^{16},$$

where m is a constant which cannot be less than a. When m is large the lines of force are practically indistinguishable from circles.

Concentric cable.

Let the inner conductor be a circular cylinder and let the outer consist of n wires each of which carries an equal share of the current. Let the distance of a point P from the axis of the inner conductor be r and from the axes of the wires be $r_1, r_2, \ldots$ respectively. The equation to the lines of force will be

$$mr^n = r_1 r_2 \ldots r_n,$$

where m is a constant.

Hence, by De Moivre's property of the circle,

$$m^2 r^{2n} = r^{2n} - 2a^n r^n \cos n\theta + a^{2n}.$$

When m equals unity, we get the curves

$$r^n \cos n\theta = \frac{a^n}{2}.$$

It is easy to see that this represents an open symmetrical curve round each wire, the vertex of which is directed towards the centre, and the equation to the asymptotes is

$$\theta = \frac{2p+1}{2n}\pi,$$

where p is an integer. When n is even, r is imaginary when θ lies between $\frac{\pi}{2n}$ and $\frac{3\pi}{2n}$, $\frac{5\pi}{2n}$ and $\frac{7\pi}{2n}$, etc. When n is odd, it is negative between these values.

When m is less than unity, $\cos n\theta$ cannot be less than $\sqrt{1-m^2}$ and hence, in this case, for each value of m we get a loop round each wire, the loops getting narrower and shorter as m diminishes.

When m is greater than unity we get rippled lines of force which only embrace the central wire. The equation to one of those passing through the point

$$\theta = \frac{\pi}{n}, \quad r = b,$$

is

$$\left\{\left(\frac{b^n + a^n}{b^n}\right)^2 - 1\right\} r^{2n} + 2a^n r^n \cos n\theta - a^{2n} = 0.$$

It follows from this equation that the minimum value of r for a given value of b is

$$\frac{ab}{\{a^n + 2b^n\}^{\frac{1}{n}}}.$$

Since b is less than a this is approximately equal to

$$b\left(1-\frac{2}{n}\frac{b^n}{a^n}\right).$$

Hence when n is large and b is small we see that the amplitude of the ripples is very small.

When b equals a, the minimum value of r is $a/3^{\frac{1}{n}}$. If there were eight wires (Fig. 121) then the maximum and minimum

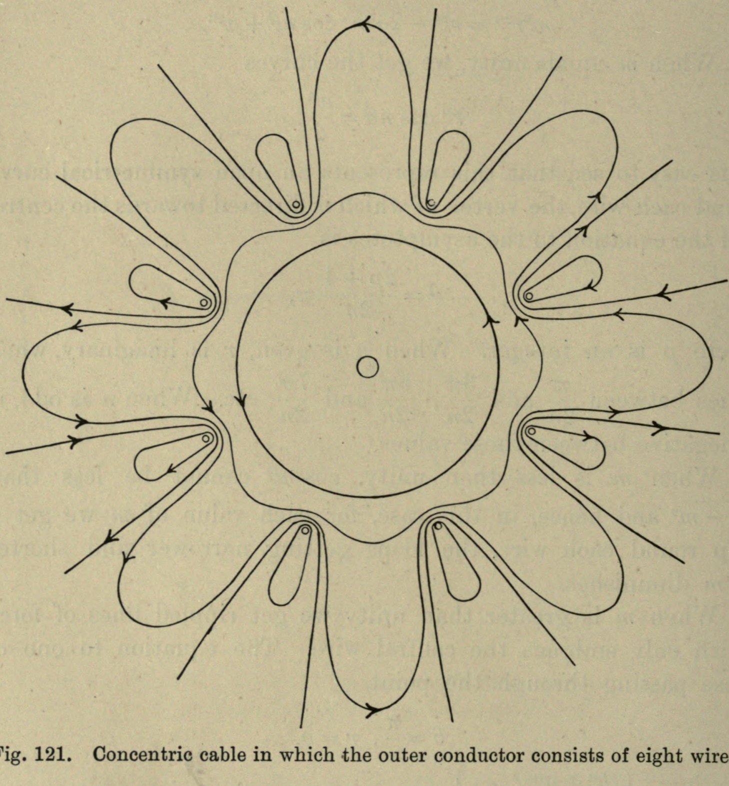

Fig. 121. Concentric cable in which the outer conductor consists of eight wires.

values of r for this line of force would be a and $0{\cdot}87a$ respectively. Hence it would differ from the circle $r = 0{\cdot}935a$ by less than 7 per cent. When b is greater than a then the minimum value of b is approximately

$$\frac{a}{2^{\frac{1}{n}}}\left(1-\frac{a^n}{2nb^n}\right),$$

which is always less than $a/2^{\frac{1}{n}}$. Hence, the lines of force which are close to the point

$$r = a/2^{\frac{1}{n}}, \ \theta = 0,$$

but pass inside it, have very large ripples, which extend in some cases to great distances beyond the circle $r = a$.

Some of the lines of force for a concentric cable having eight wires symmetrically arranged for its outer conductor are shown in Fig. 121. If it were carrying direct current, the needle of a little compass would change its direction $2n$ times on being taken round the cable. Since, with alternating currents, the current in the inner conductor is always in exact opposition in phase to the currents in any of the outer wires, the magnetic field at any point in the neighbourhood is in this case a purely oscillatory one.

In general, if we have a series of parallel wires carrying alternating currents of any magnitudes which are always in step with one another, that is, which are either in phase or in exact opposition in phase, then the magnetic field in their neighbourhood is purely oscillatory.

The strength of the magnetic field round n parallel wires.

The strength of the magnetic field in the two preceding problems is easily found by the following method, due to G. F. C. Searle, which furnishes an example of the use of Maxwell's Vector Potential.

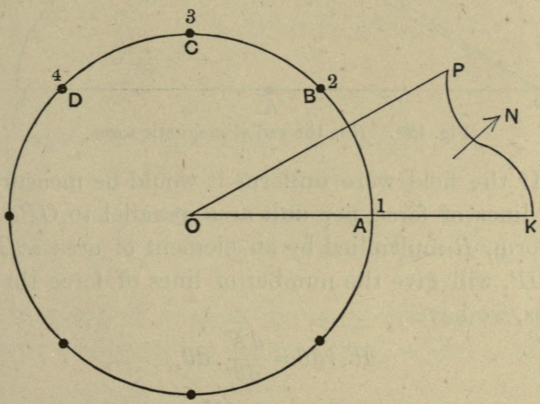

Fig. 122. *A*, *B*, *C*, *D*... are the points where the axes of n long parallel wires cut at right angles the plane of the paper.

Let us first consider the case when the magnetic force due to the current in the return conductor is negligible. Let the current in each of the n wires be i/n and let A, B, C... (Fig. 122) be the points where the axes of the wires cut the plane of the paper.

Consider two lines through P and through a fixed point K, parallel to the wires, and let N be the number of lines of force which pass between these two lines per unit length. Then, if $PA = r_1$, $PB = r_2, \ldots$, $KA = \rho_1$, $KB = \rho_2, \ldots$, $OP = r$ and $OA = a$, we have

$$N = \frac{2i}{n}\left(\log\frac{\rho_1}{r_1} + \log\frac{\rho_2}{r_2} + \log\frac{\rho_3}{r_3} + \ldots\ldots\right)$$

$$= -\frac{2i}{n}\log(r_1 r_2 \ldots r_n) + \lambda i,$$

where λ is a constant.

Hence, by De Moivre's property of the circle,

$$N - \lambda i = -\frac{i}{n}\log(r^{2n} - 2r^n a^n \cos n\theta + a^{2n}) \ldots\ldots\ldots(\alpha).$$

Now let R (Fig. 123) denote the radial magnetic force at the

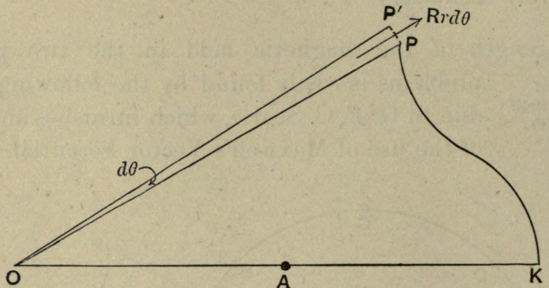

Fig. 123. R is the radial magnetic force.

point P. If the field were uniform it would be measured by the number of lines of force, per unit area, parallel to OP. When it is not uniform, R multiplied by an element of area at P at right angles to OP, will give the number of lines of force through that area. Thus, we have

$$R \,.\, rd\theta = \frac{dN}{d\theta} \,.\, d\theta,$$

and therefore $$R = \frac{1}{r}\frac{dN}{d\theta}.$$

Hence, by (α), $$R = -\frac{2i}{r} \cdot \frac{r^n a^n \sin n\theta}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} \quad \ldots(\beta).$$

Similarly, from Fig. 124, we see that the tangential force T is given by $$T = -\frac{dN}{dr}.$$

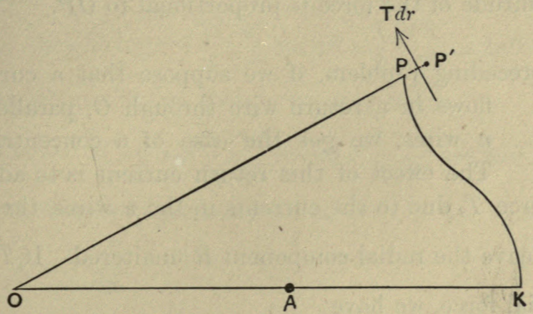

Fig. 124. ***T* is the tangential magnetic force.**

Thus, from (α), we get

$$T = \frac{2i}{r} \cdot \frac{r^{2n} - r^n a^n \cos n\theta}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} \quad \ldots\ldots\ldots\ldots\ldots(\gamma).$$

If F be the resultant force at any point in the field,

$$F^2 = R^2 + T^2,$$

since R and T are at right angles. Therefore

$$F = \frac{2i}{r} \cdot \frac{r^n}{\{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}\}^{\frac{1}{2}}}$$

$$= \frac{2i}{r} \cdot \frac{r^n}{r_1 r_2 \ldots r_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\delta).$$

Now at all points on a line of force we have

$$r_1 r_2 \ldots r_n = \text{constant},$$

and therefore, along a line of force, F will vary as r^{n-1}. For example, along any of the lines of force shown in Fig. 120 F varies as r^7.

Again, from (γ), we see that the tangential force T, that is, the force at right angles to the radius vector, is zero at all points on the curves

$$r^n = a^n \cos n\theta.$$

These are curves exactly similar to the looped curves shown in

Fig. 120, the extremities of the loops passing through the axes of the wires. If P be a point, therefore, on one of these loops, OP is the direction of the magnetic force at that point. For example, when n equals 2, we see that at any point P on the lemniscate $r^2 = a^2 \cos 2\theta$, the direction of the resultant magnetic force is OP and the magnitude of the force is proportional to OP.

The strength of the field round a concentric main.

In the preceding problem, if we suppose that a current $-i$ flows in a return wire through O, parallel to the n wires, we get the case of a concentric main. The effect of this return current is to add to the tangential force T, due to the currents in the n wires, the amount $\frac{-2i}{r}$ and to leave the radial component R unaltered. If T' be the new tangential force, we have

$$T' = T - \frac{2i}{r}$$

$$= \frac{2i}{r} \cdot \frac{r^n a^n \cos n\theta - a^{2n}}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}}.$$

This vanishes at all points on the curves

$$r^n \cos n\theta = a^n.$$

These curves are similar to the open curves shown in Fig. 121 and they pass through the axes of the wires. At any point P on these curves OP gives the direction of the magnetic force.

If F' be the resultant magnetic force at any point P,

$$F'^2 = R^2 + T'^2,$$

and thus

$$F' = \frac{2i}{r} \cdot \frac{a^n}{r_1 r_2 \ldots r_n}.$$

The magnetic force at any point on a given line of force (Fig. 121) is therefore inversely proportional to r.

Again, from (α), when r is greater than a, we get

$$N - \lambda i = -2i \log r - \frac{i}{n}\left\{\log\left(1 - \frac{a^n}{r^n}\epsilon^{n\theta\sqrt{-1}}\right) + \log\left(1 - \frac{a^n}{r^n}\epsilon^{-n\theta\sqrt{-1}}\right)\right\},$$

where $N-\lambda i$ refers to the n wires only. Thus we have

$$N-\lambda i=-2i\log r+\frac{i}{n}\left(\frac{a^n}{r^n}\epsilon^{n\theta\sqrt{-1}}+\frac{1}{2}\cdot\frac{a^{2n}}{r^{2n}}\epsilon^{2n\theta\sqrt{-1}}+\ldots\right)$$
$$+\frac{i}{n}\left(\frac{a^n}{r^n}\epsilon^{-n\theta\sqrt{-1}}+\frac{1}{2}\cdot\frac{a^{2n}}{r^{2n}}\epsilon^{-2n\theta\sqrt{-1}}+\ldots\right)$$
$$=-2i\log r+\frac{2i}{n}\left(\frac{a^n}{r^n}\cos n\theta+\frac{1}{2}\cdot\frac{a^{2n}}{r^{2n}}\cos 2n\theta+\ldots\right).$$

Similarly when r is less than a, we may write

$$N-\lambda i=-2i\log a+\frac{2i}{n}\left(\frac{r^n}{a^n}\cos n\theta+\frac{1}{2}\cdot\frac{r^{2n}}{a^{2n}}\cos 2n\theta+\ldots\right).$$

Therefore, when r is greater than a,

$$R=\frac{1}{r}\frac{dN}{d\theta}$$
$$=-\frac{2i}{r}\left(\frac{a^n}{r^n}\sin n\theta+\frac{a^{2n}}{r^{2n}}\sin 2n\theta+\ldots\right),$$

and $$T'=-\frac{dN}{dr}-\frac{2i}{r}$$
$$=\frac{2i}{r}\left(\frac{a^n}{r^n}\cos n\theta+\frac{a^{2n}}{r^{2n}}\cos 2n\theta+\ldots\right).$$

Similarly when r is less than a,

$$R=-\frac{2i}{r}\left(\frac{r^n}{a^n}\sin n\theta+\frac{r^{2n}}{a^{2n}}\sin 2n\theta+\ldots\right),$$

and $$T'=-\frac{2i}{r}-\frac{2i}{r}\left(\frac{r^n}{a^n}\cos n\theta+\frac{r^{2n}}{a^{2n}}\cos 2n\theta+\ldots\right).$$

The above series show clearly the degree of approximation of the field to that due to a longitudinal current uniformly distributed over the surface of a hollow cylinder and returning by a thin coaxial solid conductor.

The losses in cables.

If we have an alternating current, of effective value A, flowing in a conductor of resistance R, then RA^2 is the least possible value of the mean power lost in the conductor. Even when the conductor is the inner conductor of a concentric main so that it is shielded from the inductive effects of the return current, yet owing to the skin effect (p. 47), the power expended in it is greater than RA^2. In polyphase cables the wires that form the cores are not shielded from the inductive effects of the currents in neighbouring wires, or of the currents in neighbouring cores and so, except in the special case when the cores are made up of fine wires not too tightly pressed together, the eddy current losses in them are appreciable. Let us first consider the simple problem of single phase mains.

When the consumer's terminals are connected with the mains of an alternating current supply company by two conductors run through two separate iron pipes, it is found that, at times of heavy load, there is a great diminution of the voltage between the consumer's terminals. The presence of the iron magnifies very considerably the back electromotive force due to the inductance of the conductors, and thus the potential difference V_1 between the consumer's terminals is much smaller than the voltage V between the mains. In this case the potential difference V_2 between the ends of a conductor joining a main to a house terminal, that is, the 'voltage drop' along the conductor, will be large. In general the phase difference between the voltage drop and the current in a conductor is large. It will only be small in the exceptional case when the eddy current and hysteresis loss is large.

If the load in the house circuit consist of incandescent lamps so that its power factor is practically unity and if, as is generally the case in practice, the connecting conductors be equal and the losses in them are the same and take place in the same manner, we can easily measure W, the power lost in the conductors and in the pipes surrounding them. Since the voltage drop along one conductor is in phase with that along the other their resultant value is $2V_2$ and so the power W is given by the three voltmeter formula (p. 205). We have, therefore,

$$W = \frac{A}{2V_1}\{V^2 - V_1^2 - (2V_2)^2\},$$

where A is the reading of an alternating current ammeter in the circuit. If we subtract $2A^2R$ from W, where R is the resistance of either of the connecting conductors, we get the power expended in hysteresis and eddy currents in the pipes.

As V, V_1 and V_2 differ in phase, $V - V_1$ is generally much smaller than $2V_2$. For example, we could have

$$V = 260, \quad V_1 = 240, \quad V_2 = 40, \quad A = 100.$$

In this case the value of W, in watts, is given by

$$W = \frac{100}{2 \times 240} \{260^2 - 240^2 - (2 \times 40)^2\}$$

$$= 750.$$

Since W must be positive we see that when V is 260 and V_1 is 240, V_2 must be less than 50. This follows since

$$260^2 - 240^2 = (2 \times 50)^2.$$

It has been noticed that when the iron pipes are insulated from the earth the voltage drop V_2 for a given current A is less than when the pipes are laid in the earth. This shows that the eddy currents are increased when the pipe is earthed throughout its length.

To avoid this excessive drop in the pressure, the two conductors are put into the same pipe. This is found, in practice, to be a satisfactory solution. Let us suppose that the axes of the conductors and the axis of the pipe are in one plane and that these axes are parallel. Let us also suppose that the pipe is of very thin metal so that the distortion of the field produced by the eddy currents in it can be neglected. The lines of force will then practically coincide with the circles shown in Fig. 114, and an inspection of this figure will show that the plane containing the straight lines of force bisects the pipe. If the pipe be long, then at any instant the induced current will be flowing down one half of the pipe and back along the other half.

The magnetic force at any point P can be easily found. Let N be the number of lines of force per unit length of the system between P and a point on the sheath equidistant from the two wires. Then, as on p. 310, we can easily show that

$$N = -i \log (r^2 - 2ar \cos \theta + a^2) + i \log (r^2 + 2ar \cos \theta + a^2),$$

where i is the current in either conductor, $2a$ the distance between the axes of the conductors and (r, θ) are the polar coordinates of the point P, the origin being on the axis of the pipe and the initial line passing through the axis of a conductor. Thus, if R

and T be the radial and tangential components of the magnetic force at P, we get

$$R = \frac{1}{r} \cdot \frac{dN}{d\theta}$$

$$= -\frac{2i}{r} \cdot \frac{2ar \sin\theta\,(r^2 + a^2)}{r^4 - 2a^2r^2 \cos 2\theta + a^4},$$

and

$$T = -\frac{dN}{dr}$$

$$= \frac{2i}{r} \cdot \frac{2ar \cos\theta\,(r^2 - a^2)}{r^4 - 2a^2r^2 \cos 2\theta + a^4}.$$

Hence also the resultant force F is given by

$$F^2 = R^2 + T^2,$$

and thus

$$F = \frac{4ai}{r_1 r_2},$$

a result which agrees with the formula given on p. 306.

For the following method of considering the eddy current losses in a pipe, when it is insulated from earth, the author is indebted to G. F. C. Searle. We have seen that the number N of the lines of force between a point $P(r, \theta)$ on the pipe and a point equidistant from the axes of the two conductors is given by

$$N = i \log(r^2 + 2ar\cos\theta + a^2) - i \log(r^2 - 2ar\cos\theta + a^2),$$

when the walls of the pipe are very thin and the metal of which it is made is non-magnetic. Using the method employed on p. 332 and noting that r is greater than a, we have

$$N = 2i\left\{\frac{a}{r}\cos\theta - \frac{1}{2}\frac{a^2}{r^2}\cos 2\theta + \dots\right\}$$

$$+ 2i\left\{\frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}\cos 2\theta + \dots\right\}$$

$$= 4i\left\{\frac{a}{r}\cos\theta + \frac{1}{3}\frac{a^3}{r^3}\cos 3\theta + \frac{1}{5}\frac{a^5}{r^5}\cos 5\theta + \dots\right\}.$$

Thus since, by symmetry, the current vanishes at the points on the pipe equidistant from the two wires, if follows that if u be the current density at P and σ the resistivity of the pipe, we have, making the assumption that every tube of eddy current is in quadrature with the flux it embraces,

$$\sigma u = -\frac{dN}{dt}$$

$$= -4\frac{di}{dt}\left\{\frac{a}{r}\cos\theta + \frac{1}{3}\frac{a^3}{r^3}\cos 3\theta + \dots\right\}.$$

If i equal $I \cos \omega t$, the mean value of $(di/dt)^2$ is $\frac{1}{2}\omega^2 I^2$ and thus the mean value of $\sigma^2 u^2$ is

$$8\omega^2 I^2 \frac{a^2}{r^2} \left\{\cos\theta + \frac{1}{3}\frac{a^2}{r^2}\cos 3\theta + \ldots\right\}^2.$$

Hence, if the thickness of the sheath be h, we find that the mean power W lost, per unit length of the sheath, is given by

$$W = \frac{8\omega^2 I^2 a^2 h}{r^2 \sigma} \int_0^{2\pi} \left\{\cos\theta + \frac{1}{3}\frac{a^2}{r^2}\cos 3\theta + \ldots\right\}^2 r d\theta$$

$$= \frac{16\pi\omega^2 A^2 a^2 h}{r\sigma}\left\{1 + \frac{1}{9}\frac{a^4}{r^4} + \frac{1}{25}\frac{a^8}{r^8} + \ldots\right\},$$

where A is the effective value of the current. Thus, if $(a^2/3r^2)^2$ be negligible compared with unity, the eddy current loss W in the pipe, in ergs per second, is given by

$$W = (4A a\omega)^2 \frac{\pi h}{r\sigma} l,$$

where l is the length of the pipe, and all the quantities are measured in absolute units. We see that, when the reactance of every eddy current path is negligible compared with its resistance, the eddy current loss, per unit length, varies as the square of the current, the square of the distance between the axes of the wires and the square of the frequency. It also varies directly as the thickness of the pipe and inversely as its radius and its resistivity.

Let us now consider the hysteresis loss in a very thin iron pipe. If r be the radius of the pipe, the magnetic force F at a point P on its circumference is given by

$$F = \frac{4ai}{r_1 r_2}$$

$$= \frac{4ai}{(r^4 - 2a^2r^2\cos 2\theta + a^4)^{\frac{1}{2}}},$$

and thus by p. 12,

$$F = \frac{4ai}{r^2}\left\{1 + \cos 2\theta \frac{a^2}{r^2} + \left(\frac{3}{2}\cos^2 2\theta - \frac{1}{2}\right)\frac{a^4}{r^4} + \ldots\right\}.$$

It can be shown that this series is convergent when r is greater than a. F has its maximum value when θ is zero and its minimum value when θ is $\frac{\pi}{2}$. Thus the amplitudes of F in C.G.S.

measure at points in the pipe lie between

$$\frac{4I_{\text{max.}}\,a}{r^2-a^2} \text{ and } \frac{4I_{\text{max.}}\,a}{r^2+a^2},$$

where $I_{\text{max.}}$ denotes the maximum value of the current i. If we know Steinmetz's coefficient for the iron (p. 37) and the permeability curve we can easily find a superior limit to the hysteresis loss.

Three core cables.

The following data concerning three core cables and the method of their manufacture will show the nature of the problems that arise in practical work. The cores consist of stranded conductors; a 19/14 core, for example, would be one made up of 19 strands of wire of No. 14 Standard Wire Gauge, that is, wire with a diameter of 0·080 of an inch. Strips of paper are wound round the cores, and the cores are spiralled relatively to one another making a complete turn round the axis in from four to eight feet, the space between the three cores being filled either with jute or with a paper core. The cores are now wrapped together with more strips of paper, and after immersion in a bath of a special compound, are passed to the lead covering press. In the low temperature process, the lead covering is put on under very considerable pressure and at a temperature well below the melting point of lead. In most cases a covering of compounded tape or jute yarn is laid on (served) over the lead by a special machine. The armouring comes over the compounded tape and generally consists of galvanised iron wires laid on longitudinally with sufficient 'spiral' to keep them together. In some cases this armouring is served with compounded jute. The cross section of a 19/14 core is nearly 0·1 of a square inch, so that if we assume that a current density of 800 amperes per square inch is permissible, then each core can carry 80 amperes.

In some cases the armouring consists of steel ribbon or strip. This strip is put on in two layers, the first layer being spiralled in one direction and the next in the opposite direction, thus 'breaking joint' and affording efficient protection for the cable. This is the standard armouring for ordinary direct current cables, but reasons will be given later on why it should not be used for three core cables, unless the hysteresis losses in it when subjected to appreciable magnetising forces are negligible.

The minimum distance of the cores from one another and from the sheath depends on the voltage for which the cable is designed. For voltages up to 5000 it is customary to allow a thickness of 0·05 of an inch of dielectric for every thousand volts. For higher pressures the thicknesses used in practice are less than that given by this rule.

If the cable be for 11,000 volts mesh working, the minimum distance between the cores is generally about 0·4 of an inch. The inner diameter of the lead sheath would be slightly less than 2·5 inches if the cores were 0·2 of a square inch in section and would be about 3 inches if the cores were 0·4 of a square inch in section. The thickness of the lead is generally about 0·16 of an inch (160 mils). The jute between the lead and the armouring is about 50 mils in thickness but in some cables the armouring is placed directly on the lead. When steel strip is used it is generally about 40 mils thick and as there are two strips one over the other this gives a steel cylinder 80 mils thick surrounding the cable and having a diameter somewhat greater than three inches for large cables. If this be covered with jute the thickness of the jute will be about 100 mils (0·254 cm.).

The section of the cores, in the smallest of the three core cables used by the Underground Electric Railway Company of London, is 0·15 of a square inch and, in the largest, 0·25 of a square inch. The effective voltage between any pair of cores is 11,000 and the minimum distance between the cores is 0·44 of an inch. The coefficient of self induction for electrostatic charges of the cores in the smallest size of cable is 0·26 of a microfarad per mile, and in the cable which has cores 0·25 of a square inch in section the coefficients of the cores are about 0·3 of a microfarad per mile. The cables are plain lead covered.

On the power station circuit of the Metropolitan Electric Tramways Company of London, the three core cables are sheathed under the lead with copper tape as an earthing shield for protective purposes, and are drawn into earthenware ducts. The cores of most of the cables are 0·1 of a square inch in section, and the minimum distance between the cores is 0·38 of an inch. The coefficients of the cores in this class of cable are about 0·27 of a microfarad per mile.

The following are the principal data for a three core cable, for 11,000 volt mesh working, made by the British Insulated Wire Company. The section of each core is roughly similar to the segment of a circle which has a vertical angle of 120 degrees. There are 37 strands of wire in a core, the diameter of each wire being 0·082 of an inch. Each core is wrapped round with paper to a radial depth of 0·18 of an inch and the minimum distance between the cores is about 0·37 of an inch. Over this another layer of paper 0·18 inch thick is wrapped. The radial thickness of the lead sheath is 0·16 of an inch and its outer diameter is 2·5 inches. The lead in this case is served with compounded jute and over this sixty galvanised iron wires are laid, each 0·104 of an inch in diameter. Only sufficient 'spiral' is given to the wires to keep them together. Wire armouring is never put on by coiling the wire round the cable.

The weight of each of the copper cores in the above cable is about 4000 lbs. per mile. The weight of the lead sheath is over thirteen tons per mile and the weight of the wire armouring is about 10,000 lbs. per mile. The coefficient of self induction for electrostatic charges of each core is approximately 0·37 of a microfarad per mile.

In many situations, as for example in mines, mechanical protection is necessary for three core cables. They are therefore either drawn into iron pipes or heavily armoured. When this is done a copper shield under the lead is in general unnecessary.

In addition to the eddy current losses in the cores themselves and in the lead sheath, there may be considerable hysteresis and eddy current losses in the armouring. It is important therefore to know the value of the magnetic forces due to the currents at points near a three phase cable.

Formulae for the magnetic forces due to three phase cables.

We shall now find the direction and the magnitude of the magnetic force produced at any point by the currents in the cores of a three phase cable. We shall assume that the load is balanced, and that the cores can be regarded as cylinders whose axes are parallel.

Our formulae will be approximately true when the conductors are spiralled. Let i_1, i_2, and i_3 be the currents in the cores. With the notation of p. 330, if we choose the centre of the cable as K,

$$\begin{aligned} N &= 2i_1 \log a + 2i_2 \log a + 2i_3 \log a \\ &\quad - 2i_1 \log r_1 - 2i_2 \log r_2 - 2i_3 \log r_3 \\ &= -2i_1 \log r_1 - 2i_2 \log r_2 - 2i_3 \log r_3, \end{aligned}$$

since $i_1 + i_2 + i_3 = 0$.

Thus, we get

$$\begin{aligned} N &= -2(i_1 + i_2 + i_3) \log r \\ &\quad - i_1 \log \left\{1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right\} \\ &\quad - i_2 \log \left\{1 - 2\frac{a}{r}\cos\left(\theta - \frac{2}{3}\pi\right) + \frac{a^2}{r^2}\right\} \\ &\quad - i_3 \log \left\{1 - 2\frac{a}{r}\cos\left(\theta - \frac{4}{3}\pi\right) + \frac{a^2}{r^2}\right\} \end{aligned}$$

by p. 58
$$\begin{aligned} &= 2i_1 \left\{\frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}\cos 2\theta + \frac{1}{3}\frac{a^3}{r^3}\cos 3\theta + \dots\right\} \\ &\quad + 2i_2 \left\{\frac{a}{r}\cos\left(\theta - \frac{2}{3}\pi\right) + \frac{1}{2}\frac{a^2}{r^2}\cos 2\left(\theta - \frac{2}{3}\pi\right) + \dots\right\} \\ &\quad + 2i_3 \left\{\frac{a}{r}\cos\left(\theta - \frac{4}{3}\pi\right) + \frac{1}{2}\frac{a^2}{r^2}\cos 2\left(\theta - \frac{4}{3}\pi\right) + \dots\right\}. \end{aligned}$$

From this equation R and T can be found at once by the formulae

$$R = \frac{1}{r}\frac{dN}{d\theta} \quad \text{and} \quad T = -\frac{dN}{dr}.$$

In order to simplify the formulae for the radial and tangential forces we shall suppose that the currents follow the harmonic law, so that we can write

$$i_1 = I\cos\omega t, \quad i_2 = I\cos(\omega t - \tfrac{2}{3}\pi) \quad \text{and} \quad i_3 = I\cos(\omega t - \tfrac{4}{3}\pi).$$

Now, we have

$$\begin{aligned} &2\cos\omega t \cos p\theta \\ &\qquad = \cos(\omega t + p\theta) + \cos(\omega t - p\theta), \\ &2\cos(\omega t - \tfrac{2}{3}\pi)\cos p(\theta - \tfrac{2}{3}\pi) \\ &\qquad = \cos\{\omega t + p\theta - \tfrac{2}{3}(p+1)\pi\} + \cos\{\omega t - p\theta + \tfrac{2}{3}(p-1)\pi\}, \end{aligned}$$

and

$$2\cos(\omega t - \tfrac{4}{3}\pi)\cos p(\theta - \tfrac{4}{3}\pi)$$
$$= \cos\{\omega t + p\theta - \tfrac{4}{3}(p+1)\pi\} + \cos\{\omega t - p\theta + \tfrac{4}{3}(p-1)\pi\}.$$

If S be the sum of these quantities, we have

$$S = \frac{\cos\{\omega t + p\theta - \frac{2}{3}(p+1)\pi\}\sin(p+1)\pi}{\sin\frac{1}{3}(p+1)\pi}$$
$$+ \frac{\cos\{\omega t - p\theta + \frac{2}{3}(p-1)\pi\}\sin(p-1)\pi}{\sin\frac{1}{3}(p-1)\pi}.$$

Now, the value of $\sin x\pi / \sin \frac{1}{3}x\pi$ is zero, when x is neither zero nor an integral multiple of 3. As x approaches any integral multiple of 3 the quantity approaches the limit 3. Using these results we obtain

$$N = 3I\left\{\frac{a}{r}\cos(\omega t - \theta) + \frac{1}{2}\frac{a^2}{r^2}\cos(\omega t + 2\theta)\right.$$
$$+ \frac{1}{4}\frac{a^4}{r^4}\cos(\omega t - 4\theta) + \frac{1}{5}\frac{a^5}{r^5}\cos(\omega t + 5\theta)$$
$$\left. + \frac{1}{7}\frac{a^7}{r^7}\cos(\omega t - 7\theta) + \dots\dots\dots\right\}.$$

Hence we find the following value for R

$$R = \frac{1}{r}\frac{dN}{d\theta}$$
$$= \frac{3I}{r}\left\{\frac{a}{r}\sin(\omega t - \theta) - \frac{a^2}{r^2}\sin(\omega t + 2\theta)\right.$$
$$+ \frac{a^4}{r^4}\sin(\omega t - 4\theta) - \frac{a^5}{r^5}\sin(\omega t + 5\theta)$$
$$\left. + \dots\dots\dots\dots\dots\dots\right\}.$$

Putting their exponential values for $\sin\theta$, $\cos\theta$, $\sin 2\theta$, etc. we get simple geometrical progressions for the coefficients of $\sin\omega t$ and $\cos\omega t$. Summing the series and replacing the exponential values by sines and cosines we find that

$$R = \frac{3I}{a}\cdot\frac{a^2\{(r^4 + a^4)\sin(\omega t - \theta) - ar(r^2 + a^2)\sin(\omega t + 2\theta)\}}{r^6 - 2a^3r^3\cos 3\theta + a^6}.$$

It can be shown also, that this formula is true when r is equal to or less than a. Again, we have

$$T = -\frac{dN}{dr}$$

$$= \frac{3I}{r}\left\{\frac{a}{r}\cos(\omega t - \theta) + \frac{a^2}{r^2}\cos(\omega t + 2\theta) + \frac{a^4}{r^4}\cos(\omega t - 4\theta) + \frac{a^5}{r^5}\cos(\omega t + 5\theta) + \ldots\ldots + \ldots\ldots\right\}$$

$$= \frac{3I}{a}\cdot\frac{a^2\{(r^4 - a^4)\cos(\omega t - \theta) + ar(r^2 - a^2)\cos(\omega t + 2\theta)\}}{r^6 - 2a^3r^3\cos 3\theta + a^6}.$$

This formula is also true when r is less than or equal to a. We see at once that T vanishes when r equals a. The amplitude of T equals

$$\frac{3I}{a}\cdot\frac{a^2(r^2 - a^2)\{(r^2 + a^2)^2 + 2ar(r^2 + a^2)\cos 3\theta + a^2r^2\}^{\frac{1}{2}}}{r^6 - 2a^3r^3\cos 3\theta + a^6}.$$

For a given value of r this has obviously its greatest value $T_{\text{max.}}$ when $\cos 3\theta$ is 1 and its least value $T_{\text{min.}}$ when $\cos 3\theta$ is -1. Thus we have

$$T_{\text{max.}} = \frac{3I}{a}\cdot\frac{a^2(r+a)}{r^3 - a^3},$$

and

$$T_{\text{min.}} = \frac{3I}{a}\cdot\frac{a^2(r-a)}{r^3 + a^3}.$$

It is easy to show that $T_{\text{min.}}$ has a maximum value when r is $1{\cdot}679\,a$ and $T_{\text{min.}}$ is then equal to $\frac{3I}{a} \times 0{\cdot}1184$.

An inspection of the following table will show how the coefficients of $\frac{3I}{a}$ in the formulae for $T_{\text{max.}}$ and $T_{\text{min.}}$ vary as r increases.

$\frac{r}{a}$	1	1·5	2	2·5	3	4	10	100
$\frac{a^2(r+a)}{r^3-a^3}$	∞	1·053	0·429	0·239	0·154	0·079	0·011	0·000
$\frac{a^2(r-a)}{r^3+a^3}$	0	0·114	0·111	0·090	0·071	0·046	0·009	0·000

The formulae for R and T given above show us that at all points which are outside the three cores the field is elliptical. Along the axis of the cable the ellipse is a circle, at points on the circle passing through the axes of the three cores T vanishes and thus the ellipse becomes a straight line pointing to the centre of the circle, and when a^2/r^2 can be neglected in comparison with a/r we again get a pure rotating field the strength of which is $3aI/r^2$.

If O be the centre of the section of the cable, A the point where the axis of the core cuts this section and P a point on OA or OA produced, the values of R and T at P are given by

$$R = \frac{3I}{a} \cdot \frac{a^2(r-a)}{r^3 - a^3} \sin \omega t,$$

and

$$T = \frac{3I}{a} \cdot \frac{a^2(r+a)}{r^3 - a^3} \cos \omega t$$

$$= T_{\text{max.}} \cos \omega t.$$

Thus at all points on OA or OA produced the elliptical field has its major axis perpendicular to OA and equal to $2T_{\text{max.}}$.

Similarly, if OP' bisect the angle BOA where B is a point on the axis of another core, the field at a point P on OP', at a distance r from O, is given by

$$R = \frac{3I}{a} \cdot \frac{a^2(r+a)}{r^3 + a^3} \sin \left(\omega t - \frac{\pi}{3}\right),$$

and

$$T = \frac{3I}{a} \cdot \frac{a^2(r-a)}{r^3 + a^3} \cos \left(\omega t - \frac{\pi}{3}\right)$$

$$= T_{\text{min.}} \cos \left(\omega t - \frac{\pi}{3}\right).$$

The minor axis of the ellipse representing the field at P equals $2T_{\text{min.}}$ and its direction is perpendicular to OP.

The following table shows how the coefficients of $3I/a$ in the expression for the radial force vary as we move in the directions OA or OP.

$\frac{r}{a}$	1	1·5	2	2·5	3	4	10	100
$\frac{a^2(r-a)}{r^3-a^3}$	0·333	0·211	0·143	0·103	0·077	0·048	0·009	0·000
$\frac{a^2(r+a)}{r^3+a^3}$	1	0·571	0·333	0·211	0·143	0·077	0·011	0·000

If f be the value of the resultant force at the point (r, θ) we have

$$f^2 = R^2 + T^2$$

$$= \left(\frac{3I}{a}\right)^2 \cdot \frac{a^4\{a^2 + r^2 + 2ar\cos(2\omega t + \theta)\}}{r^6 - 2a^3r^3\cos 3\theta + a^6},$$

which agrees with the result obtained on p. 317. Since the maximum value of $\cos 3\theta$ is 1 and its minimum value is -1, we find that the maximum and minimum values of f at points at a distance r from the axis of the cable are given by

$$f_{\text{max.}} = \frac{3I}{a} \cdot \frac{a^2(r+a)}{r^3 - a^3} = T_{\text{max.}},$$

and

$$f_{\text{min.}} = \frac{3I}{a} \cdot \frac{a^2(r-a)}{r^3 + a^3} = T_{\text{min.}}.$$

Let us now consider the hysteresis loss in the steel armouring of a three core cable. We shall suppose that the armour consists of very thin steel strip, and we shall neglect the shielding effect of the eddy currents induced in the lead sheath. The integral of the tangential force at any instant taken round the steel cylinder is zero, the lines of induction therefore leave the cylinder and demagnetising effects are produced. If we calculate the hysteresis loss on the assumption that there are no demagnetising effects and that $T_{\text{max.}}$ acts on every point of the armouring we get a superior limit to the hysteresis losses. If A be the effective value of the alternating current in a core, expressed in amperes, $T_{\text{max.}}$ will be given in C.G.S. units by the formula

$$T_{\text{max.}} = 3\,\frac{A\sqrt{2}}{10a} \cdot \frac{a^2(r+a)}{r^3 - a^3}.$$

As an example we shall take the case of a three core cable designed for 10,000 volt working and having a normal current of about 150 amperes in each core. The area of the cross section of each core is 1·25 square centimetres, the radius a of the circle circumscribing the axes of the cores 1·8 cms. and the inner radius of the steel armouring 4·5 cms. If the effective current in each core be A amperes, the maximum value of the magnetic force in C.G.S. units is $0·056\,A$. If, therefore, A were 150 amperes, the maximum value of the magnetic force would be 8·4 C.G.S. units. Hence, if we know the permeability curve and Steinmetz's coefficient for the steel (p. 37), we can assign a superior limit to the hysteresis loss.

We have seen that the minimum value of the tangential force at a distance r from the axis is given by the formula

$$T_{\text{min.}} = 3\frac{A\sqrt{2}}{10a}\cdot\frac{a^2(r-a)}{r^3+a^3}.$$

For the cable considered above we get

$$\begin{aligned} T_{\text{min.}} &= 0{\cdot}021\,A \\ &= 3{\cdot}2 \text{ C.G.S. units,} \end{aligned}$$

when A is 150 amperes. This, however, does not enable us to fix a minimum value to the tangential force acting on the steel as we cannot neglect the considerable demagnetising effect produced by the lines of induction leaving the steel.

If we make the assumption that the mean value of the maximum flux density is 1000 and take $\eta = 0{\cdot}002$, we see, by p. 38, that the loss W, in watts per kilogramme of the steel, when the frequency is 50, is given by

$$\begin{aligned} W &= \eta f V B_{\text{max.}}^{1{\cdot}6} \times 10^{-7} \\ &= 0{\cdot}002 \times 50 \times \frac{1000}{7{\cdot}8} \times (1000)^{1{\cdot}6} \times 10^{-7} \\ &= 0{\cdot}08 \text{ nearly,} \end{aligned}$$

assuming that the specific gravity of steel is 7·8. If the steel armouring therefore have a mass of 2000 kilogrammes per mile this would give a loss of 0·16 of a kilowatt per mile.

When iron wire is used, with its length parallel to the axis of the cable, the magnetic flux produced will be very much smaller than with steel strip owing to the very large demagnetising effects produced by the flux leaving the wires. The hysteresis loss with wire armouring will therefore be quite negligible.

In addition to the hysteresis losses, there are eddy current losses in the cores themselves, in the lead sheath, in the copper earth shield and in the armouring. We shall see in the next chapter that the currents generated near the surface of the sheath screen off the magnetic force from the interior, and thus in the simplest cases even with non-magnetic metals, the calculation of the eddy current losses is difficult. It has to be remembered, in estimating possible eddy current losses in sheaths, that these losses do not vary in any simple manner with the resistivity of the metal of which the sheath is made.

In addition to the losses that take place in the metallic parts of the cable, we may have losses in the dielectric itself due to the electrostatic forces. When the cores are very thin it is not difficult to find formulae for the resultant electrostatic force at points in the dielectric by the method of images (Chapter V). Let us consider, for example, the electrostatic force f at the axis of the cable. We suppose that all the electrical quantities are given in electrostatic units. The component f_1 of the force in the direction OA, due to the wire whose axis is at A and the image of this wire at A', is given by (Chapter IV),

$$\lambda f_1 = -\frac{2q_1}{a} + \frac{2q_1}{a'},$$

where $OA = a$, and $OA' = a'$ and λ is the dielectric coefficient. Now when the load is balanced (Chapter IV), so that

$$v_1 + v_2 + v_3 = 0,$$

then

$$q_1 = (K_{1.1} - K_{1.2})\, v_1,$$

and thus, since f_1, f_2 and f_3 are inclined to each other at angles of 120°, we get by Statics

$$f^2 = f_1^2 + f_2^2 + f_3^2 - f_2 f_3 - f_3 f_1 - f_1 f_2,$$

and hence, on substituting, we get

$$\lambda^2 f^2 = 4\left(\frac{1}{a} - \frac{1}{a'}\right)^2 (K_{1.1} - K_{1.2})^2 \{v_1^2 + v_2^2 + v_3^2 - v_2 v_3 - v_3 v_1 - v_1 v_2\}$$

$$= 6\left(\frac{1}{a} - \frac{a}{r^2}\right)^2 (K_{1.1} - K_{1.2})^2 (v_1^2 + v_2^2 + v_3^2),$$

since $aa' = r^2$, where r is the inner radius of the sheath.

When the potential differences follow the harmonic law, we get

$$\lambda f = 3\left(\frac{1}{a} - \frac{a}{r^2}\right)(K_{1.1} - K_{1.2})\, E,$$

where E is the maximum value of v_1 and therefore of the star pressure.

Let us take the case of a cable working with an effective pressure of 11,000 volts between the cores. Let $a = 1{\cdot}5$, $r = 4{\cdot}5$

and let the capacity between two of the cores be 0·15 of a microfarad per mile. Then, by p. 92,

$\frac{1}{2}(K_{1.1} - K_{1.2}) = 0\text{·}15$ of a microfarad per mile, and thus

$$(K_{1.1} - K_{1.2})\, E = 0\text{·}3 \times 10^{-15} \times 11{,}000\, \sqrt{\tfrac{2}{3}} \times 10^{8}$$

c.g.s. electromagnetic units per mile

$$= 0\text{·}3 \times 10^{-7} \times 11{,}000 \sqrt{\frac{2}{3}} \times \frac{3 \times 10^{10}}{160{,}900}$$

$$= 50\text{·}2 \text{ c.g.s. electrostatic units per centimetre,}$$

and thus $\lambda f = 89\text{·}3$.

If we assume that λ is 2·8 (see p. 108) we get

$$f = 31\text{·}9 \text{ dynes per unit charge nearly.}$$

This corresponds to a rate of variation of the potential, at the axis of the cable, in the direction of f, equal to $31\text{·}9 \times 300 \times 2\text{·}8$, that is, 26,800 volts per centimetre. If the frequency of the alternating current be 25 then f will make 25 revolutions per second. As the force is appreciable, experiments are being made to determine whether the magnitudes of the electrostatic losses are large enough to be measurable.

It follows from the formulae we have found for the magnetic force near a three phase cable with straight cores, that there will be an appreciable disturbance set up in telephone circuits if the wires are near the cable and are not twisted together. When the cable is armoured the disturbance will be less. As the armouring is always thin compared with the diameter of the cable and the radial magnetic forces are appreciable, the diminution of the disturbance due to the armouring may, however, only be slight. The intensity of the disturbance at distances greater than a metre from the cable is approximately inversely proportional to the square of the distance provided that the going and return wires of the telephone circuit be close together.

The formulae we have found for the magnetic force round three core cables also apply to the forces round overhead wires carrying three phase currents, provided that the wires are equidistant from one another. In this case, if the telephone wires be carried on the same posts as the wires for the three phase currents and be not twisted together, a continual hum will be heard in the telephone. This will be slight if the mains are spiralled, and can be made negligible by 'crossing' the telephone wires at regular intervals, that is, making the higher wire the lower wire for the next section and so on. In the event of a short circuit occurring at an insulator and earthing one of the mains, the currents in them will no longer be balanced, and a loud hum will be set up in the telephone. J. Behn-Eschenburg has used the telephone as a fault signaller for several years on the 30,000 volt transmission line at the Oerlikon Machine Works in Switzerland.

Duality.

The equations and the diagrams for the lines of force and equipotential lines given in this chapter are similar to the equations and the diagrams for the equipotential lines and the lines of force in electrostatics. This is an example of the principle of duality (Chapter XVII). The equipotential line in electrostatics corresponds to the magnetic line of force, and the electrical line of force corresponds to the line of equal magnetic potential (see p. 138). The lines of force in statical electricity are also the same curves as the lines of flow of an electric current through a conducting medium. The diagrams of the lines of force given above can be obtained experimentally by maintaining suitably chosen spots on a sheet of tinfoil at given potentials and mapping out the equipotential lines on the sheets by Carey Foster's method.

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CHAPTER XVI.

Eddy currents. Short circuited coil in a uniform alternating magnetic field. Currents in the closed secondary of a constant voltage transformer. Eddy currents in the iron plates of the core of a transformer. Eddy currents produced in an infinite iron plate, placed in a uniform harmonically varying magnetic field, with its sides parallel to the lines of force. Analogy with the theory of heat. Approximate formulae. Eddy currents induced in a metallic cylinder forming the core of a long solenoid. Tables of cosh θ, sinh θ, cos θ, sin θ, ϵ^n, ber (x), bei (x), ber$'$ (x), and bei$'$ (x). References.

Eddy currents.

THE calculation of the losses due to the eddy currents, which are generated in conductors when placed in variable magnetic fields, is a problem of great practical importance. In most cases only roughly approximate solutions can be obtained, owing to the difficulty of solving the equations even when the conductors are of non-magnetic materials and the strength of the field varies harmonically. Oliver Heaviside has found a solution for the case of a solid cylinder of infinite length, whose axis is parallel to the lines of force of a uniform alternating magnetic field, and J. J. Thomson has solved the corresponding problem for the case of an infinite plate when the surface of the plate is parallel to the lines of force of the field. Before giving an elementary discussion of these problems we shall consider the problem of the currents induced in a closed coil of insulated wire, when placed in an alternating magnetic field.

Short circuited coil in a uniform alternating magnetic field.

Let us suppose that the coil has n turns of wire, that the mean area enclosed by the turns is S and that the induction density parallel to the axis of the coil is

$$B \sin \omega t.$$

Let R be the resistance of the coil, L its self inductance and i the induced current, then

$$Ri + L\frac{di}{dt} = -n\frac{d}{dt}(SB\sin\omega t)$$
$$= -nSB\omega\cos\omega t.$$

The effective value A of the current is therefore given by

$$A = \frac{nSB\omega}{\sqrt{2}\sqrt{R^2 + L^2\omega^2}}.$$

Hence for a given maximum induction density B, the effective value of the current will continually increase as ω increases and therefore as the frequency increases, but it can never be greater than $\frac{nSB}{L\sqrt{2}}$. The power expended in heating the coil is RA^2 and this equals

$$\frac{(nSB\omega)^2 R}{2(R^2 + L^2\omega^2)}.$$

This expression continually increases as the frequency increases and its limiting value is $\left(\frac{nSB}{L\sqrt{2}}\right)^2 R$. If we suppose that R is the variable, then the heating for a given frequency will be a maximum when R equals $L\omega$ or $2\pi fL$. Hence, the increase of the resistivity of the metal of the conductor, due to the heating, will increase the power expended in heating the circuit if R is less than $2\pi fL$ but will diminish it if R is greater than $2\pi fL$.

Currents in the closed secondary of a constant voltage transformer.

From the equations of the air core transformer it is easy to show (Chapter X), that

$$-\frac{M}{L_1}(e_1 - R_1 i_1) = R_2 i_2 + L_2\sigma\frac{di_2}{dt},$$

where

$$\sigma = 1 - \frac{M^2}{L_1 L_2}$$
$$= \text{the leakage factor.}$$

Thus, squaring and taking mean values, we find that

$$\frac{M^2}{L_1^2}(V_1^2 - 2R_1 W + R_1^2 A_1^2) = R_2^2 A_2^2 + \alpha^2 L_2^2 \sigma^2 \omega^2 A_2^2,$$

where α is a constant which has its minimum value unity (p. 80) when e_1 follows the harmonic law. Now W is the mean value of the power taken by the primary coil. It must therefore be equal to the mean rate of heat production in both circuits and thus

$$W = R_1 A_1^2 + R_2 A_2^2.$$

Hence, on substituting this value of W in the above equation, we get

$$A_2^2 \left(R_2^2 + 2 \frac{M^2}{L_1^2} R_1 R_2 + \alpha^2 L_2^2 \sigma^2 \omega^2 \right) = \frac{M^2}{L_1^2} (V_1^2 - R_1^2 A_1^2).$$

The power expended on the secondary circuit is therefore

$$\frac{R_2}{R_2^2 + 2 \frac{M^2}{L_1^2} R_1 R_2 + \alpha^2 L_2^2 \sigma^2 \omega^2} \cdot \frac{M^2}{L_1^2} (V_1^2 - R_1^2 A_1^2).$$

Assuming that $R_1 A_1$ is negligible compared with V_1, we see that, when σ is not zero, the power expended in heating the secondary circuit continually diminishes as the frequency increases and vanishes when the frequency is infinite.

If the secondary resistance vary, then, if we neglect $R_1 A_1$ compared with V_1, the power transmitted to the secondary circuit at a given frequency is a maximum when R_2 equals $2\pi f \alpha L_2 \sigma$ and it then equals

$$\frac{1}{2 \frac{M^2}{L_1^2} R_1 + 2R_2} \left(\frac{M}{L_1} V_1 \right)^2,$$

provided that the shape of the current wave in the secondary circuit remain constant. When R_2 is greater than $2\pi f \alpha L_2 \sigma$, an increase in the resistance of the circuit diminishes the loss due to the heating in the circuit, but if R_2 is less than $2\pi f \alpha L_2 \sigma$ then an increase of R_2 increases the loss.

When $R_1 A_1$ is small compared with V_1 and the leakage factor is zero, the heating of the secondary circuit is simply equal to

$$\frac{1}{R_2 + 2 \frac{M^2}{L_1^2} R_1} \left(\frac{M}{L_1} V_1 \right)^2,$$

and it is therefore independent of the frequency and of the shape of the applied potential difference wave. When the leakage factor is not zero the heating of the secondary circuit for a given effective voltage V_1 is a maximum when the applied potential difference wave is sine-shaped.

The above theorems for insulated fine wire coils, although instructive, do not give us much help in calculating the eddy current losses in masses of metal subjected to an alternating magnetising force. To do this we must find the paths and the magnitudes of the induced currents in the metal.

Eddy currents in the iron plates of the core of a transformer.

It is found in practice that, unless the iron plates which form the core of a transformer are less than about half a millimetre in thickness, the heating due to eddy currents is appreciable. The cores are therefore built up of finely laminated iron. The lamination appears to have little, if any, effect on the losses due to hysteresis. In the core of a transformer the plates are practically thin strips of iron, the length of the strips being parallel to the direction of the alternating magnetic field. The induced electromotive force will therefore be parallel to the cross section of the strips, and if their thickness be small compared with their breadth, we can suppose that the lines of flow are at right angles to the length of the strips and parallel to the faces of the strips. These currents will alter the value of the magnetising force H, and hence H will have different values at different distances from the surface of the plate. We shall show that there is a diminution in the amplitude of H as we go into the plate. To compensate for this diminution in the value of the magnetising force, we must increase the magnetising current of the transformer, and hence increase the losses in the copper conductors. Again, since by Steinmetz's law the hysteresis loss varies as $B^{1 \cdot 6}$, the hysteresis loss will be greater than if the magnetic induction were uniform and had its density equal to the mean value of B. The iron at the centre of the plates is screened, to a certain extent, by the eddy currents, and we shall see that this screening effect is greater the thicker the plates and the higher the frequency of the magnetic force.

Eddy currents produced in an infinite iron plate, placed in a uniform harmonically varying magnetic field, with its sides parallel to the lines of force.

Let AA' (Fig. 125) be a line perpendicular to the two faces of the plate and let O be the middle point of AA'. Let OZ be parallel to the direction of the applied magnetic force and let OY be at right angles to OZ and OX. Suppose that the field is produced by harmonic alternating currents flowing in a coil of insulated wire wrapped round the plate, the windings lying in planes parallel to the plane XOY, the coil forming an infinitely long and infinitely broad solenoid with OZ for its axis, so that the lines of force are everywhere parallel to OZ. By making the breadth of the strip great enough in proportion to its thickness, we can consider that the eddy currents, except near the edges, are all flowing in directions parallel to OY. We suppose therefore that every tube of current which cuts the plane XOZ at a distance x from OZ also cuts the same plane at a distance $-x$ from it.

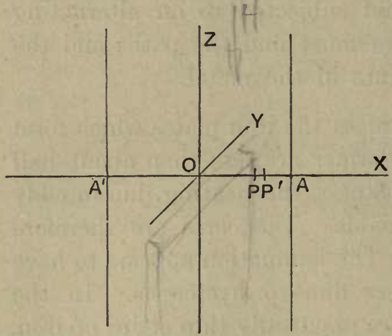

Fig. 125. AA' the thickness of the plate equals $2a$, and YOZ is the median plane.

Let i be the current density at a point P (Fig. 125) in the iron, where OP equals x, and let σ be the resistivity of iron. Consider a small circuit in the plane XOY of breadth dx and unit length. The electromotive force round this circuit is $\sigma\left(i+\frac{di}{dx}dx\right)-\sigma i$, and this, by Faraday's law, must be equal to the rate at which the lines of induction linked with the circuit diminish. Thus, if B denote the induction density at the point P at the time t, we have

$$\sigma\left(i+\frac{di}{dx}dx\right)-\sigma i=-\frac{d}{dt}(Bdx),$$

and thus $$\sigma\frac{di}{dx}=-\frac{dB}{dt} \quad\text{..................(1).}$$

Let us now consider a small circuit in the plane XOZ. Suppose that its breadth is dx and its length parallel to OZ is unity. The current flowing across this section parallel to OY will be idx. Hence, equating the two expressions for the work done in taking a unit magnetic pole round the boundary of this circuit, we obtain

$$4\pi\,.\,idx=H-\left(H+\frac{dH}{dx}dx\right).$$

Therefore $$i=-\frac{1}{4\pi}\frac{dH}{dx} \quad\text{..........................(2).}$$

Now the induction density B is not in simple proportion to the magnetising force H. Owing to hysteresis, B is different when H is increasing or diminishing, and the ratio of B to H is also different for different values of H. To take hysteresis into account in our mathematical equations would therefore be very difficult. If we neglect hysteresis we may write μH for B, where μ is a single valued function of H. Hence from (1) and (2) we get

$$\frac{d}{dt}(\mu H) = \frac{\sigma}{4\pi}\frac{d^2H}{dx^2}.$$

If we make the further assumption that μ is constant for the range of forces with which we are concerned in practice, then

$$\frac{dH}{dt} = \frac{\sigma}{4\pi\mu}\frac{d^2H}{dx^2} \quad \text{.........................(3).}$$

Analogy with the theory of heat.

This equation is identical in form with the equation for the linear diffusion of heat through a conducting plate, namely

$$\frac{dv}{dt} = \frac{k}{c}\frac{d^2v}{dx^2},$$

where v is the temperature at a distance x from a central plane, k is the thermal conductivity of the substance and c is its thermal capacity per unit volume. Comparing the electrical and thermal equations, we see that resistivity corresponds to thermal conductivity and permeability to thermal capacity. An infinitely good electrical conductor corresponds to a thermal insulator, and a substance having high permeability to a substance having great capacity for heat. The magnetic force will diffuse into the metal according to the same law as heat diffuses into it, provided the applied magnetic forces at the faces of the plate vary according to the same law as the temperatures of the faces (see also Chapter V). It can be shown that this theorem is true in all cases. Hence we can make use of the known solutions of problems in the theory of conduction of heat to find the values of H and therefore of i in the corresponding electrical problems.

Let the thickness of the plate be $2a$ and let the value of H at the faces of the plate be $H_0 \cos \omega t$. Then H must be a function of x and t which satisfies the differential equation (3) and has the value $H_0 \cos \omega t$, when x is $+a$ or $-a$.

Let us suppose that

$$H = h \cos \omega t + k \sin \omega t$$

is a solution of (3) where h and k are functions of x.

Substituting this value for H in (3), we get

$$\omega(-h \sin \omega t + k \cos \omega t) = \frac{\sigma}{4\pi\mu}\left(\frac{d^2h}{dx^2}\cos \omega t + \frac{d^2k}{dx^2}\sin \omega t\right),$$

a relation which must be true for all values of t. Hence we must have

$$h = -\frac{\sigma}{4\pi\mu\omega}\frac{d^2k}{dx^2}, \quad \text{and} \quad k = \frac{\sigma}{4\pi\mu\omega}\frac{d^2h}{dx^2},$$

and therefore

$$h = -\left(\frac{\sigma}{4\pi\mu\omega}\right)^2 \frac{d^4h}{dx^4}$$

$$= -\frac{1}{4m^4}\frac{d^4h}{dx^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where

$$m^2 = \frac{2\pi\mu\omega}{\sigma}$$

$$= (2\pi)^2 \frac{\mu f}{\sigma},$$

where f is the frequency of the magnetising current.

Thus

$$m = 2\pi\sqrt{\frac{\mu f}{\sigma}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now assume that h is $A\epsilon^{nx}$ and substitute this value in (4). We find that

$$n^4 = -4m^4,$$

therefore

$$n^2 = 2m^2(\pm\sqrt{-1}),$$

and

$$n = \pm m(1 \pm \sqrt{-1}).$$

The complete solution of (4) is therefore

$$h = A\epsilon^{mx}\cos mx + B\epsilon^{mx}\sin mx + C\epsilon^{-mx}\cos mx + D\epsilon^{-mx}\sin mx,$$

where A, B, C and D are constants.

Again, we have

$$k = \frac{1}{2m^2}\frac{d^2h}{dx^2}.$$

On substituting for h and performing the differentiations, we get

$$k = -A\epsilon^{mx}\sin mx + B\epsilon^{mx}\cos mx + C\epsilon^{-mx}\sin mx - D\epsilon^{-mx}\cos mx.$$

Now $$H = h\cos\omega t + k\sin\omega t,$$
and therefore
$$H = A\epsilon^{mx}\cos(\omega t + mx) + B\epsilon^{mx}\sin(\omega t + mx)$$
$$+ C\epsilon^{-mx}\cos(\omega t - mx) - D\epsilon^{-mx}\sin(\omega t - mx).$$
Again, since, by symmetry, H does not change sign with x for any value of t, we see, by putting ωt equal to zero and equal to $\frac{\pi}{2}$ successively, that neither h nor k must change sign with x. It will be seen from inspection that
$$A = C \quad\text{and}\quad B = -D$$
satisfy both these conditions. We get, therefore,
$$H = A\left\{\epsilon^{mx}\cos(\omega t + mx) + \epsilon^{-mx}\cos(\omega t - mx)\right\}$$
$$+ B\left\{\epsilon^{mx}\sin(\omega t + mx) + \epsilon^{-mx}\sin(\omega t - mx)\right\} \ldots\ldots(6).$$
Now at the faces of the plate H equals $H_0\cos\omega t$ and thus,
$$h = H_0 \quad\text{and}\quad k = 0, \quad\text{when}\quad x = \pm a.$$
We get, therefore
$$H_0 = A(\epsilon^{ma} + \epsilon^{-ma})\cos ma + B(\epsilon^{ma} - \epsilon^{-ma})\sin ma,$$
$$0 = A(\epsilon^{ma} - \epsilon^{-ma})\sin ma - B(\epsilon^{ma} + \epsilon^{-ma})\cos ma.$$
Writing $\cosh ma$ for $\frac{1}{2}(\epsilon^{ma} + \epsilon^{-ma})$ and $\sinh ma$ for $\frac{1}{2}(\epsilon^{ma} - \epsilon^{-ma})$ and solving the equations we find that
$$\left.\begin{aligned} A &= \frac{H_0\cosh ma\cos ma}{\cosh 2ma + \cos 2ma} \\ \text{and}\qquad B &= \frac{H_0\sinh ma\sin ma}{\cosh 2ma + \cos 2ma}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(7).$$
Again, we have
$$h = 2A\cosh mx\cos mx + 2B\sinh mx\sin mx,$$
and therefore
$$\frac{h}{H_0} = \frac{\cosh m(a-x)\cos m(a+x) + \cosh m(a+x)\cos m(a-x)}{\cosh 2ma + \cos 2ma},$$
and $k = -2A\sinh mx\sin mx + 2B\cosh mx\cos mx,$
hence
$$\frac{k}{H_0} = \frac{\sinh m(a-x)\sin m(a+x) + \sinh m(a+x)\sin m(a-x)}{\cosh 2ma + \cos 2ma}.$$

Now, $$H = h\cos\omega t + k\sin\omega t$$

$$= \sqrt{h^2 + k^2}\cos\left(\omega t - \tan^{-1}\frac{k}{h}\right),$$

and $$\frac{h^2+k^2}{H_0^2} = 4\frac{A^2+B^2}{H_0^2}(\cosh^2 mx\cos^2 mx + \sinh^2 mx\sin^2 mx)$$

$$= \frac{4(\cosh^2 ma\cos^2 ma + \sinh^2 ma\sin^2 ma)(\cosh^2 mx\cos^2 mx + \sinh^2 mx\sin^2 mx)}{(\cosh 2ma + \cos 2ma)^2}$$

$$= \frac{(\cosh 2ma + \cos 2ma)(\cosh 2mx + \cos 2mx)}{(\cosh 2ma + \cos 2ma)^2}$$

$$= \frac{\cosh 2mx + \cos 2mx}{\cosh 2ma + \cos 2ma}.$$

Thus, on substituting, we get

$$H = H_0\left(\frac{\cosh 2mx + \cos 2mx}{\cosh 2ma + \cos 2ma}\right)^{\frac{1}{2}}\cos(\omega t - \gamma)\quad\text{.........(8)},$$

where $\tan\gamma = \frac{k}{h}$

$$= \frac{\sinh m(a-x)\sin m(a+x) + \sinh m(a+x)\sin m(a-x)}{\cosh m(a-x)\cos m(a+x) + \cosh m(a+x)\cos m(a-x)}\quad\text{...(9)}.$$

Now, since

$$\cosh 2mx + \cos 2mx = 2\left\{1 + \frac{(2mx)^4}{\underline{|4}} + \frac{(2mx)^8}{\underline{|8}} + \text{.........}\right\},$$

we see that, as x increases from zero, $\cosh 2mx + \cos 2mx$ increases continually from its least value 2. Thus, we see from (8) that the amplitude of H diminishes as we approach the centre of the plate, where it has its minimum value which is given by (11) below. We see also from (9) that the phase of H is different for different values of x. If the thickness of the plate were $\frac{2n\pi}{m}$, that is, if ma equals $n\pi$, then $\sin mx$ would be a factor of both terms in the numerator and γ would be zero when x is $0, \frac{\pi}{m}, \frac{2\pi}{m}, \ldots \frac{n\pi}{m}$. Hence at depths $\frac{2\pi}{m}, \frac{4\pi}{m}, \ldots$ the magnetising force, in this case, would be in phase with its surface value.

If γ_c be the phase difference between the phase of the surface value of H and the value of H at the centre, then by (9)

$$\tan\gamma_c = \tan ma\tanh ma\quad\text{..................(10)}.$$

Also, if H_c be the amplitude of H at the centre, then

$$H_c = \frac{H_0\sqrt{2}}{(\cosh 2ma + \cos 2ma)^{\frac{1}{2}}} \quad \text{...............(11).}$$

In the sheet-iron used in the cores of transformers ordinary values of σ and μ are 10,000 C.G.S. units and 2500 respectively. If the frequency were 100, then by (5)

$$m = 2\pi\sqrt{\frac{2500 \times 100}{10,000}},$$

$$= 10\pi.$$

Therefore if ma equal π, that is, if the thickness of the plates were two millimetres, we see from (10) that γ_c would be 180 degrees and from (11) by the aid of the table on p. 376 that H_c would be $H_0/11{\cdot}6$. If the plates were four millimetres thick γ_c would be 360 degrees, that is, the values of H at the surface and the centre of the plate would be in phase with one another but H_c would only be $H_0/268$. If the thickness of the plates were one millimetre, then γ_c would be ninety degrees and H_c would be $H_0/2{\cdot}4$.

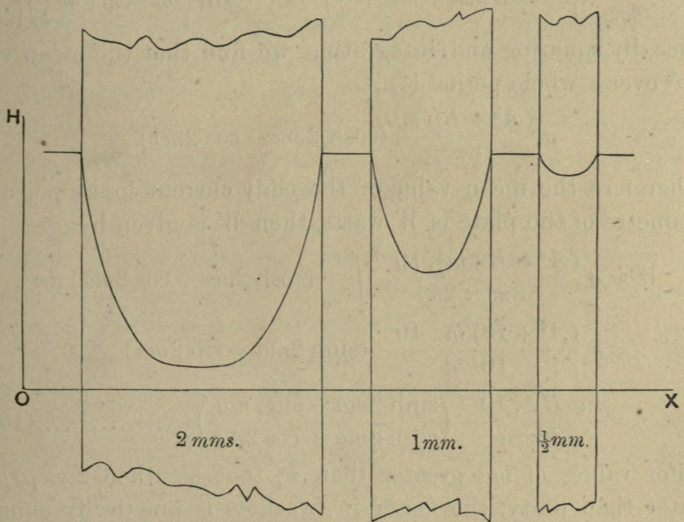

Fig. 126. The amplitudes of H at various depths in iron sheets which are 2 mms., 1 mm. and ½ mm. thick respectively.
$\mu = 2500$. $\sigma = 10{,}000$. $f = 100$.

In Fig. 126 the relative values of the maximum amplitudes of H, when the frequency of the alternations is 100, at various depths in iron sheets 2, 1 and $\frac{1}{2}$ a millimetre thick respectively are shown. The diagrams do not show instantaneous values of H. For example, in the two millimetre plate, when H has its maximum positive value at the surface it has its maximum negative value at the centre. The permeability of the iron is supposed to be 2500 and its resistivity 10,000 C.G.S. units. The same result would follow if the permeability were 400 and the resistivity 1600 absolute units, if the frequency be still 100. Increasing the permeability or diminishing the resistivity increases the screening effect of the eddy currents. The diagrams in Fig. 126 show that we should not use iron sheets thicker than half a millimetre for the cores of transformers when the frequency is 100.

From (6), since $i = -\frac{1}{4\pi}\frac{dH}{dx}$, we find that

$$i = -\frac{Am\sqrt{2}}{4\pi}\left\{\epsilon^{mx}\cos\left(\omega t + mx + \frac{\pi}{4}\right) - \epsilon^{-mx}\cos\left(\omega t - mx + \frac{\pi}{4}\right)\right\}$$
$$- \frac{Bm\sqrt{2}}{4\pi}\left\{\epsilon^{mx}\sin\left(\omega t + mx + \frac{\pi}{4}\right) - \epsilon^{-mx}\sin\left(\omega t - mx + \frac{\pi}{4}\right)\right\}.$$

Hence, by squaring and integrating, we find that the mean value of σi^2 over a whole period is

$$\sigma\frac{(A^2+B^2)\,m^2}{8\pi^2}(\cosh 2mx - \cos 2mx).$$

If therefore the mean value of the eddy current losses per cubic centimetre of the plate is W watts, then W is given by

$$W = \sigma\frac{(A^2+B^2)\,m^2 . 10^{-7}}{8\pi^2 \times 2a}\int_{-a}^{+a}(\cosh 2mx - \cos 2mx)\,dx$$
$$= \sigma\frac{(A^2+B^2)\,m . 10^{-7}}{16\pi^2 a}(\sinh 2ma - \sin 2ma)$$
$$= \frac{m\sigma H_0^2 . 10^{-7}}{32\pi^2 a}\left(\frac{\sinh 2ma - \sin 2ma}{\cosh 2ma + \cos 2ma}\right) \quad \text{...............(12).}$$

For values of ma greater than π, that is, when $2a\sqrt{\mu f/\sigma}$ is greater than unity, the factor in brackets is practically equal to unity, and the formula becomes

$$W = \frac{\sqrt{\mu f \sigma}H_0^2 . 10^{-7}}{16\pi a}.$$

We see, therefore, that quadrupling the resistivity σ, provided that $2a\sqrt{\mu f/\sigma}$ still remained greater than unity, would double the eddy current loss.

When ma is small, γ_c is nearly zero and H_c is nearly equal to H_0. Hence the amplitude of H is approximately constant across the plate. Hence, if $B_{\text{max.}}$ be the maximum density of the flux, we may write

$$B_{\text{max.}} = \mu H_0.$$

Since ma is small, we see from (12) that

$$W = \frac{m\sigma H_0^2 . 10^{-7}}{32\pi^2 a} \cdot \frac{\frac{1}{6}(2ma)^3}{1+\frac{1}{24}(2ma)^4}$$

$$= \frac{m^4\sigma a^2 H_0^2 . 10^{-7}}{24\pi^2} \text{ nearly}$$

$$= \frac{\pi^2}{6}\frac{(2a)^2}{\sigma} f^2 B^2_{\text{max.}} . 10^{-7}$$

$$\doteq 1{\cdot}64 \frac{(2a)^2}{\sigma} f^2 B^2_{\text{max.}} . 10^{-7}.$$

When the iron, subjected to the harmonic magnetising forces, consists of thin sheets, this formula gives us the loss of power in watts expended by eddy currents per cubic centimetre of the iron. The thickness $2a$ of each sheet must be expressed in centimetres, the resistivity σ is in C.G.S. units (about 10,000), f is the frequency and $B_{\text{max.}}$ is the maximum value of the flux density. In proving this formula we have supposed that the iron is free from hysteresis and that μ is constant.

We have seen that, when the iron sheets are thick, H varies both in amplitude and phase at different depths in the iron. In finding the flux we must therefore take this into account.

The mean value of the magnetising force for all the points on a line, inside the metal, perpendicular to the faces of the plate will be a harmonic function of the time. Hence, if $H_{\text{max.}}$ be the maximum value of the mean magnetic force, we can write

$$H_{\text{max.}} \sin(\omega t + \alpha) = \frac{1}{2a}\int_{-a}^{+a} H dx$$

$$= \frac{1}{a}\int_0^a H dx.$$

Substituting the value of H got from (6) and performing the integration, we get, after simplifying,

$$H_{\text{max.}} = \frac{H_0}{\sqrt{2} \, . \, am} \left(\frac{\cosh 2ma - \cos 2ma}{\cosh 2ma + \cos 2ma} \right)^{\frac{1}{2}} \quad \text{........(13).}$$

Now, if a depth d on each side of the plate were uniformly magnetised by a force H_0 and if the sum of the fluxes produced in these portions of the plate were equal to the maximum value of the total flux produced in the actual plate by the given magnetising forces, then

$$d \, . \, H_0 = a \, . \, H_{\text{max.}},$$

and hence
$$d = \frac{1}{m\sqrt{2}} \left(\frac{\cosh 2ma - \cos 2ma}{\cosh 2ma + \cos 2ma} \right)^{\frac{1}{2}} \quad \text{............(14).}$$

Ewing calls d the 'equivalent depth of uniform magnetisation.' The maximum value of the total flux produced in a thick iron plate placed in an alternating magnetic field, whose value is given by $H_0 \sin \omega t$, is equal to the total flux which would be produced in two layers of depth d on each side of it uniformly magnetised by the force H_0.

From (14) we see that when $2ma$ is large the equivalent depth d_1 is given by

$$d_1 = \frac{1}{m\sqrt{2}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\sigma}{2\mu f}}.$$

When $2ma$ is less than $\frac{\pi}{2}$, d is less than d_1. When it is greater than $\frac{\pi}{2}$ and less than $\frac{3\pi}{2}$, d is greater than d_1. Hence we see that d is alternately less and greater than d_1 as the thickness of the plate is increased, the amplitude of the oscillations of d about the value d_1 continually diminishing. Whenever $2ma$ equals $(2n+1)\frac{\pi}{2}$, where n is an integer, d equals d_1. Suppose, for example, that μ is 5000, f is 50 and σ is 10,000, then

$$m = 2\pi \sqrt{\frac{\mu f}{\sigma}} = 10\pi.$$

The following table is calculated using this value of m.

$2a$ in millimetres	0·25	0·50	0·75	1	1·5	2	∞
$2d$ in millimetres	0·248	0·450	0·514	0·491	0·450	0·449	0·450

In order to calculate the hysteresis loss in the iron sheets, we need to know the amplitude of H at various depths in the iron. Ewing has shown that the hysteresis loss in iron sheets, for values of B between 2000 and 8000, may be expressed approximately by a simple empirical formula of the form

$$W' = \alpha H - \beta,$$

where W' gives the loss in ergs per cubic centimetre per cycle due to hysteresis. For example, in a particular sample he found

$$W' = 1340H - 1610.$$

Now if we could find the mean value of the maximum amplitudes of H at all points on a line in the metal perpendicular to the faces of the plate, then this formula would enable us to find the hysteresis loss in the plate. If we make the assumption that the eddy currents are the same when hysteresis is present, we can find the mean value of H by the formula (8) given above. To do this by the integral calculus is difficult, but it can easily be done graphically. We have only to plot a curve, having the amplitude of H for ordinate and the depth (x) for abscissa, similar to those shown in Fig. 126. The required mean value will then be $\frac{1}{a}\int_0^a y dx$, that is, the area of the curve divided by its breadth. The area can easily be found by a planimeter.

The curves in Fig. 127 show graphically the relative values of hysteresis and eddy current losses in a transformer core for various thicknesses of the iron plates, obtained by making the given assumptions. The losses for the various thicknesses of the plates were calculated by Ewing by the methods described above. It is interesting to note that on the given assumptions the hysteresis loss increases with the thickness of the plates owing to the fact that it varies as $B^{1·6}_{\text{max.}}$, and hence it will be greater than if B were constant and equal to its mean value. The diagram also shows that, when the frequency is 100, the eddy current losses become appreciable when the thickness of the plates is greater than a quarter of a millimetre.

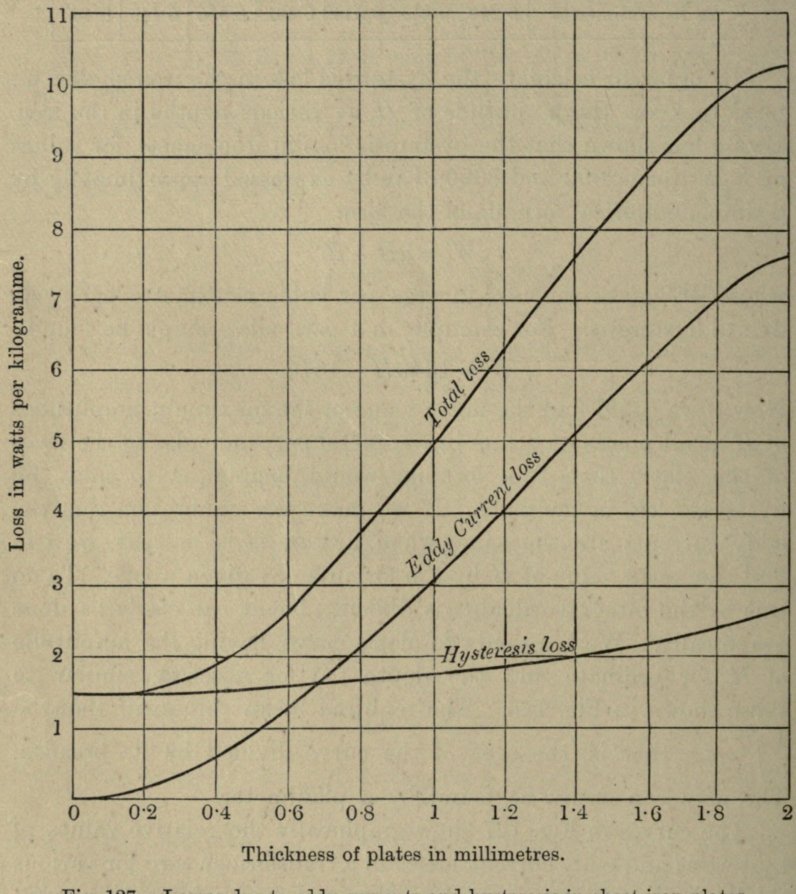

Fig. 127. Losses due to eddy currents and hysteresis in sheet-iron plates when B is 4000 and the frequency is 100.

As a first rough approximation it is permissible to apply the formulae given above to iron sheets, but as we have neglected the effects of hysteresis in modifying the eddy currents, they may be affected by large errors. An inspection of the curves given on p. 35 will show that the ratio of B to H is far from being constant in practice, and that when H follows the harmonic law B follows some other and more complicated law.

Let us suppose that the amplitude of the magnetic force is constant over the cross section of a thin sheet of iron (Fig. 125), and that its phase is also constant. Let it be represented by $H_0 \cos \omega t$. By Fourier's theorem we can also suppose that B is given by

$$B = B_1 \cos(\omega t - \alpha_1) + B_3 \cos(3\omega t - \alpha_3) + \ldots.$$

The intensity i of the current in the metal at a distance x from the central plane, is given by

$$\sigma i = \frac{d}{dt}(Bx),$$

and therefore
$$\sigma i^2 = \frac{x^2}{\sigma}\left(\frac{dB}{dt}\right)^2.$$

The mean rate at which heat is being produced at any moment, per unit volume of the sheet,

$$= \frac{1}{2\sigma a}\left(\frac{dB}{dt}\right)^2 \int_{-a}^{a} x^2\, dx,$$

$$= \frac{a^2}{3\sigma}\left(\frac{dB}{dt}\right)^2.$$

Thus the average rate of heat production, per unit volume of the plate,

$$= \frac{a^2}{3\sigma}\frac{1}{T}\int_0^T \omega^2 \{-B_1 \sin(\omega t - \alpha_1) - 3B_3 \sin(3\omega t - \alpha_3) - \ldots\}^2\, dt$$

$$= \frac{a^2}{6\sigma}\omega^2 \{B_1^2 + (3B_3)^2 + (5B_5)^2 + \ldots\}$$

$$= \frac{\pi^2}{6}\frac{(2a)^2}{\sigma} f^2 \{B_1^2 + (3B_3)^2 + \ldots\},$$

where the result is given in ergs per second. When B_3, B_5, ... are all zero, this agrees with our former result. When the amplitudes of the higher harmonics cannot be neglected, our former result will only be true in the very special case when

$$B^2_{\text{max.}} = B_1^2 + (3B_3)^2 + (5B_5)^2 + \ldots.$$

The hysteresis loss W', in ergs per second per unit volume, will be given by (see p. 36)

$$W' = \frac{f}{4\pi}\int_0^T H\frac{dB}{dt}\,dt$$

$$= \frac{H_0 f}{4\pi}\int_0^T \cos\omega t\,\{-B_1\omega\sin(\omega t-\alpha_1) - 3B_3\omega\sin(3\omega t-\alpha_3) - \ldots\}\,dt$$

$$= \frac{H_0 f T}{8\pi}\,.\,B_1\omega\sin\alpha_1$$

$$= \frac{H_0 B_1 f}{4}\sin\alpha_1.$$

If η denote Steinmetz's coefficient (p. 36), we have

$$\frac{H_0 B_1}{4}\sin\alpha_1 = \eta B^{1\cdot 6}_{\text{max.}}.$$

If therefore $B_{\text{max.}}$ equal μH_0, we find that

$$B_1\sin\alpha_1 = 4\eta\mu B^{0\cdot 6}_{\text{max.}}.$$

Since the maximum value of $\sin\alpha_1$ is unity, it follows that B_1 cannot be less than $4\eta\mu B^{0\cdot 6}_{\text{max.}}$. To determine its value, we could plot a curve of B, as a function of the time, from the known hysteresis loop of the iron corresponding to $B_{\text{max.}}$ and then apply harmonic analysis. This method, however, is not accurate in practice, as small errors in drawing the curve may introduce large errors into the values of the harmonics deduced from it.

The following experimental method of finding B_1, B_3, ... and α_1, α_3, ... is much more accurate. Thin strips of the iron are bent into the shape of a ring and a primary and a secondary coil are wound round the ring as in magnetic testing (p. 34). An alternating current, the wave of which is sine shaped, is passed through the primary coil and the values of E_1, E_3, ... and α_1', α_3', ... for the wave of electromotive force induced in the secondary coil are found, with an oscillograph, by Armagnat's resonance method of analysing waves (see Vol. II, Chap. III). Now,

$$E_{2n+1}\cos\{(2n+1)\omega t-\alpha'_{2n+1}\} = -Sn_2\frac{d}{dt}\,[B_{2n+1}\cos\{(2n+1)\,\omega t-\alpha_{2n+1}\}],$$

where S is the area of the cross section of the iron core, in square centimetres, and n_2 is the number of turns in the secondary coil. We thus find that

$$B_{2n+1} = \frac{E_{2n+1}}{(2n+1)\,\omega S n_2} \quad \text{and} \quad \alpha_{2n+1} = \alpha'_{2n+1} - \frac{\pi}{2}.$$

Thus, when the amplitude of the magnetic force at all points of the core is constant and when its phase is also constant, B can be determined completely. It is worth noting that, in this case, if the resistance of the primary coil be negligible, the eddy current loss will be proportional to the square of the effective value of the applied potential difference, and will be independent of the shape of the wave of this applied voltage.

Approximate Formulae. For a copper plate, placed in a uniform alternating magnetic field, μ will be unity, and σ may be taken equal to 1600, hence

$$m = \frac{\pi}{20}\sqrt{f}$$
$$= 0{\cdot}1571\sqrt{f}.$$

When ma is large, we see from (7) that we may write

$$A = H_0 \epsilon^{-ma} \cos ma,$$
$$B = H_0 \epsilon^{-ma} \sin ma,$$

and therefore from (6)

$$H = H_0 \{\epsilon^{-m(a-x)} \cos(\omega t - m\overline{a-x}) + \epsilon^{-m(a+x)} \sin(\omega t - m\overline{a+x})\}.$$

Hence, when $a - x$ is small compared with a,

$$H = H_0 \epsilon^{-m(a-x)} \cos(\omega t - m\overline{a-x}).$$

At a depth d in the copper plate, which is small compared with its thickness $2a$, we may write,

$$H = H_0 \epsilon^{-md} \cos(\omega t - md) \quad \text{...............(15)}.$$

We see, then, that as we go into the plate the amplitude of H diminishes according to the exponential law, and also that at points whose depths differ by a multiple of $\frac{2\pi}{m}$ the values of H are in phase with one another. The maximum values of H at depths $0, \frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \dots \frac{10}{m}$ are

$$1{\cdot}00H, \quad 0{\cdot}368H_0, \quad 0{\cdot}135H_0, \quad 0{\cdot}050H_0, \dots \; 0{\cdot}000045H_0.$$

Eddy currents induced in a metallic cylinder forming the core of a long solenoid.

We suppose that the windings of the solenoid can be regarded as practically parallel to one plane, so that the magnetic field inside is parallel to the axis and has a value $4\pi nc$, at all points between the solenoid and the core, where c is the instantaneous value of the alternating magnetising current and n is the number of turns per unit length. We shall denote the radius of the core by R. The eddy currents in the core will, from symmetry, be circles whose centres are on the axis of the solenoid. Let r be the radius of a ring whose breadth is dr and thickness unity, then, if i be the intensity of the current in it, $4\pi . idr$ will be the amount by which H changes when we pass from r to $r + dr$.

Therefore

$$H - \left(H + \frac{dH}{dr} dr\right) = 4\pi i dr,$$

and thus

$$i = -\frac{1}{4\pi} \frac{dH}{dr} \qquad \text{.........................} (a).$$

The total electromotive force round the circle of radius r is $2\pi r i \sigma$ and is also equal to the rate at which the total flux of induction through it is diminishing. Hence, assuming a constant permeability, we get

$$2\pi r i \sigma = -\mu \int_0^r \frac{dH}{dt} 2\pi r dr \qquad \text{..................} (b).$$

Hence from (b) and (a)

$$-\frac{r\sigma}{4\pi} \cdot \frac{dH}{dr} = -\mu \int_0^r \frac{dH}{dt} r dr,$$

and therefore

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{dH}{dr}\right) = \frac{4\pi\mu}{\sigma} \frac{dH}{dt} \qquad \text{.....................} (c).$$

This equation is identical in form with the corresponding problem of the diffusion of heat into a cylinder. We will assume that the magnetising current follows the harmonic law so that the magnetic force at the surface of the core is $H_0 \cos \omega t$.

Since H will be a periodic function of the time, we may assume the solution of (c) to be

$$H = H_1 \cos \omega t + H_2 \sin \omega t,$$

where H_1 and H_2 are functions of r, but not of t. Substituting this value for H in (c) we get

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH_1}{dr}\right)=\frac{4\pi\mu\omega}{\sigma}H_2, \text{ and } \frac{1}{r}\frac{d}{dr}\left(r\frac{dH_2}{dr}\right)=-\frac{4\pi\mu\omega}{\sigma}H_1 \ldots(d).$$

Therefore

$$H_1=-\left(\frac{\sigma}{4\pi\mu\omega}\right)^2\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dH_1}{dr}$$

$$=-\frac{1}{m^4}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dH_1}{dr}\ldots\ldots\ldots\ldots\ldots\ldots(e),$$

where $m^2=\dfrac{4\pi\mu\omega}{\sigma}$. We see also that H_2 satisfies an exactly similar equation.

If $$H_1=a_0+a_1r+a_2r^2+a_3r^3+\ldots,$$

then, by substituting in (e), we find that

$$a_0+a_1r+a_2r^2+\ldots=-\frac{1}{m^4}\left\{\frac{a_1}{r^3}+\frac{a_3.1^2.3^2}{r}+a_4.2^2.4^2\right.$$
$$\left.+a_5.3^2.5^2r+a_6.4^2.6^2r^2+\ldots\right\}.$$

Hence $$a_1=0,\quad a_3=0,\quad a_5=0,\ \ldots,$$

and $$a_4=-\frac{m^4}{2^2.4^2}a_0,\quad a_6=-\frac{m^4}{4^2.6^2}a_2,\ \ldots.$$

Therefore

$$H_1=a_0\left(1-\frac{m^4r^4}{2^2.4^2}+\frac{m^8r^8}{2^2.4^2.6^2.8^2}-\ldots\right)$$
$$+\frac{4a_2}{m^2}\left(\frac{m^2r^2}{2^2}-\frac{m^6r^6}{2^2.4^2.6^2}+\ldots\right).$$

Kelvin writes this in the form

$$H_1=a_0\operatorname{ber}(mr)+\frac{4a_2}{m^2}\operatorname{bei}(mr),$$

where $$\operatorname{ber}(x)=1-\frac{x^4}{2^2.4^2}+\frac{x^8}{2^2.4^2.6^2.8^2}-\ldots,$$

and $$\operatorname{bei}(x)=\frac{x^2}{2^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots.$$

If $J_0(x)$ be the Bessel's function of zero order, we have

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots.$$

Hence

$$J_0\{x(-1)^{\frac{1}{4}}\} = 1 - \frac{x^2}{2^2}\sqrt{-1} - \frac{x^4}{2^2 . 4^2} + \frac{x^6}{2^2 . 4^2 . 6^2}\sqrt{-1} + \dots .$$

Therefore $\quad \text{ber}(x) - \sqrt{-1}\,\text{bei}(x) = J_0\{x(-1)^{\frac{1}{4}}\}.$

Similarly $\quad \text{ber}(x) + \sqrt{-1}\,\text{bei}(x) = J_0\{x\sqrt{-1}(-1)^{\frac{1}{4}}\}$

$$= J_0\{x(-1)^{\frac{3}{4}}\}.$$

Hence ber(x) and bei(x) can be found in terms of Bessel's functions of complex arguments and from the known properties of these functions many relations will follow. For our purpose it is sufficient to know that ber(x) and bei(x) are series which are convergent for all values of x, however great, and that the sums of these series can be calculated to any desired degree of accuracy.

Using (d) and differentiating the series for ber(mr) and bei(mr) we find that

$$H_2 = \frac{1}{m^2 r}\frac{d}{dr} r \frac{dH_1}{dr}$$

$$= -a_0 \,\text{bei}(mr) + \frac{4a_2}{m^2}\text{ber}(mr).$$

Hence $\quad H = \left\{a_0 \,\text{ber}(mr) + \frac{4a_2}{m^2}\text{bei}(mr)\right\}\cos \omega t$

$$+ \left\{\frac{4a_2}{m^2}\text{ber}(mr) - a_0\,\text{bei}(mr)\right\}\sin \omega t.$$

Now, when r equals R, the radius of the cylinder, we must have H equal to $H_0 \cos \omega t$. Hence the boundary conditions give us

$$H_0 = a_0\,\text{ber}(mR) + \frac{4a_2}{m^2}\text{bei}(mR),$$

$$0 = \frac{4a_2}{m^2}\text{ber}(mR) - a_0\,\text{bei}(mR).$$

Therefore $\quad a_0 = \dfrac{H_0\,\text{ber}(mR)}{\text{ber}^2(mR) + \text{bei}^2(mR)},$

and $\quad \dfrac{4a_2}{m^2} = \dfrac{H_0\,\text{bei}(mR)}{\text{ber}^2(mR) + \text{bei}^2(mR)}.$

The complete solution is therefore

$$H = C\{\text{ber}(mR)\,\text{ber}(mr) + \text{bei}(mR)\,\text{bei}(mr)\}\cos \omega t$$

$$+ C\{\text{bei}(mR)\,\text{ber}(mr) - \text{ber}(mR)\,\text{bei}(mr)\}\sin \omega t \dots (f),$$

where $$C = \frac{H_0}{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)}.$$

The amplitude of H, at a point at a distance r from the axis of the cylinder, is

$$H_0 \left\{\frac{\operatorname{ber}^2(mr) + \operatorname{bei}^2(mr)}{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)}\right\}^{\frac{1}{2}} \quad\text{..................}(g),$$

and we shall show later on that this increases as r increases. At the axis of the cylinder, where r equals zero, the amplitude becomes

$$\frac{H_0}{\{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)\}^{\frac{1}{2}}}.$$

If γ be the phase difference between the value of H at a distance r from the axis and its value on the circumference of the cylinder, then

$$\tan\gamma = \frac{\operatorname{bei}(mR)\operatorname{ber}(mr) - \operatorname{ber}(mR)\operatorname{bei}(mr)}{\operatorname{ber}(mR)\operatorname{ber}(mr) + \operatorname{bei}(mR)\operatorname{bei}(mr)} \quad\text{......}(h).$$

Tables for $\operatorname{ber}(x)$ and $\operatorname{bei}(x)$ are given at the end of the chapter.

When r is zero, $\operatorname{ber}(mr)$ is 1 and $\operatorname{bei}(mr)$ is 0, and then

$$\tan\gamma_c = \frac{\operatorname{bei}(mR)}{\operatorname{ber}(mR)} \quad\text{........................}(j).$$

Now $$m^2 = \frac{4\pi\mu\omega}{\sigma}$$

$$= \frac{8\pi^2\mu f}{\sigma}.$$

Hence, if the frequency of the applied magnetising force be 100 and if the cylinder be made of copper, for which μ equals unity and σ equals 1600, we have

$$m^2 = \frac{8\pi^2 . 100}{1600},$$

and therefore $$m = 2{\cdot}22 \text{ nearly}.$$

Now, from the tables, we see that $\operatorname{bei}(mR)$ vanishes when mR is zero, it then increases as mR increases and attains a maximum value for some value of mR between 3·5 and 4. It then diminishes and vanishes when mR is a little greater than 5. When mR is 5,

R is 2·2 centimetres nearly, $\tan\gamma_c$ is approximately zero and the maximum value of H along the axis of the cylinder is $H_0/6{\cdot}2$. Now, from (j) and the tables for $\text{bei}(x)$ and $\text{ber}(x)$ given at the end of the chapter, we see that when mR is zero, γ_c is zero. As mR increases $\tan\gamma_c$ increases and after attaining an infinite positive value it becomes negative and vanishes when mR is a little greater than 5. We thus see that when mR is a little greater than 5, γ_c is π and the magnetic force along the axis is in opposition in phase to the magnetic force at the surface. In this case also, the amplitude of the magnetic force along the axis is less than the sixth part of the amplitude of the magnetic force at the surface.

The rate of generation of heat at any point of the core, per unit volume, is σi^2 ergs per second. Hence for a length l of the core the instantaneous value w of the power expended in eddy currents is given by

$$w = l\int_0^R \sigma i^2 . 2\pi r\, dr.$$

But
$$i = -\frac{1}{4\pi}\frac{dH}{dr},$$

and therefore by (f)

$$-4\pi i = Cm\,\{\text{ber}\,(mR)\,\text{ber}'\,(mr) + \text{bei}\,(mR)\,\text{bei}'\,(mr)\}\cos\omega t$$
$$+ Cm\,\{\text{bei}\,(mR)\,\text{ber}'\,(mr) - \text{ber}\,(mR)\,\text{bei}'\,(mr)\}\sin\omega t,$$

where
$$\text{ber}'(x) = \frac{d}{dx}\{\text{ber}(x)\},$$

and
$$\text{bei}'(x) = \frac{d}{dx}\{\text{bei}(x)\}.$$

The effective value A of the current is therefore given by

$$32\pi^2 A^2 = H_0^2 m^2 \frac{\text{ber}'^2(mr) + \text{bei}'^2(mr)}{\text{ber}^2(mR) + \text{bei}^2(mR)},$$

and since m^2 equals $8\pi^2\dfrac{\mu f}{\sigma}$, we get

$$A^2 = \frac{\mu f}{4\sigma} H_0^2 \frac{\text{ber}'^2\,(mr) + \text{bei}'^2\,(mr)}{\text{ber}^2\,(mR) + \text{bei}^2\,(mR)}.$$

Now, if W watts be the mean power expended in a length l of the core,

$$W = 10^{-7} l \int_0^R A^2 \sigma 2\pi r\, dr,$$

and thus

$$W = \tfrac{1}{2}\pi\mu f l H_0^2 10^{-7} \frac{\int_0^R \{r \operatorname{ber}'^2(mr) + r \operatorname{bei}'^2(mr)\}\, dr}{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)}.$$

Again, integrating by parts,

$$\int \operatorname{ber}'(mr) \,.\, r \operatorname{ber}'(mr)\, dr = \frac{1}{m} \operatorname{ber}(mr) \,.\, r \operatorname{ber}'(mr)$$
$$- \frac{1}{m} \int \operatorname{ber}(mr) \frac{d}{dr} \{r \operatorname{ber}'(mr)\}\, dr,$$

and

$$\int \operatorname{bei}'(mr) \,.\, r \operatorname{bei}'(mr)\, dr = \frac{1}{m} \operatorname{bei}(mr) \,.\, r \operatorname{bei}'(mr)$$
$$- \frac{1}{m} \int \operatorname{bei}(mr) \frac{d}{dr} \{r \operatorname{bei}'(mr)\}\, dr.$$

By differentiating the series for ber (mr) and bei (mr), it is easy to show that

$$\frac{d}{dr}\, r\, \frac{d}{dr} \operatorname{ber}(mr) = -m^2 r \operatorname{bei}(mr),$$

and

$$\frac{d}{dr}\, r\, \frac{d}{dr} \operatorname{bei}(mr) = m^2 r \operatorname{ber}(mr).$$

The sum of the integrals on the right-hand sides of the two equations therefore vanishes when they are taken between the same limits, and thus, integrating between the limits, we have

$$W = \frac{1}{2m} \pi R \mu f l H_0^2 10^{-7} \,.\, \frac{\operatorname{ber}(mR) \operatorname{ber}'(mR) + \operatorname{bei}(mR) \operatorname{bei}'(mR)}{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)}.$$

Since W must always be positive we see that

$$\operatorname{ber}(mR) \operatorname{ber}'(mR) + \operatorname{bei}(mR) \operatorname{bei}'(mR),$$

or

$$\frac{1}{2m} \frac{d}{dR} \{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)\}$$

must always be positive. Thus $\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)$ increases as mR increases, and thus also, $\operatorname{ber}^2(x) + \operatorname{bei}^2(x)$ increases as x increases. This result is useful in studying some of the equations given above.

In practice we want to know the average power expended per cubic centimetre of the conductor on account of eddy currents. Hence, dividing W by $\pi l R^2$ and noting that m^2 equals $8\pi^2\mu f/\sigma$, we find that the loss is

$$\frac{\sqrt{\mu f \sigma} H_0^2 10^{-7}}{4\pi R\sqrt{2}} \cdot \frac{\operatorname{ber}(mR)\operatorname{ber}'(mR) + \operatorname{bei}(mR)\operatorname{bei}'(mR)}{\operatorname{ber}^2(mR) + \operatorname{bei}^2(mR)}$$

watts per cubic centimetre of the core.

In this formula σ, H_0 and R are in C.G.S. units. The numerical value of this expression, in any given case, can be found easily by means of the tables for ber (x), bei (x), ber$'$ (x) and bei$'$ (x) given at the end of the chapter. We have to remember however that we made two assumptions when proving this formula, namely, that the magnetising force H follows the harmonic law and that μ, the permeability of the core, is a constant.

When mR is small we have approximately

$$\operatorname{ber}(mR) = 1 - \frac{m^4R^4}{2^2 \cdot 4^2}, \qquad \operatorname{ber}'(mR) = -\frac{m^3R^3}{16},$$

$$\operatorname{bei}(mR) = \frac{m^2R^2}{2^2}, \quad \text{and} \quad \operatorname{bei}'(mR) = \frac{mR}{2}.$$

Hence, going as far as $(mR)^3$, we have, if W denote the eddy current loss per cubic centimetre,

$$W = \frac{\sqrt{\mu f \sigma} H_0^2 10^{-7}}{4\pi R\sqrt{2}} \cdot \frac{1 \times \left(-\frac{m^3R^3}{16}\right) + \frac{m^2R^2}{2^2} \times \frac{mR}{2}}{1}.$$

$$= \frac{\sqrt{\mu f \sigma} H_0^2 10^{-7}}{4\pi R\sqrt{2}} \cdot \frac{R^3}{16}\left(\frac{8\pi^2\mu f}{\sigma}\right)^{\frac{3}{2}}$$

$$= \frac{\pi}{4} H_0^2 (\pi R^2) \frac{\mu^2 f^2}{\sigma} 10^{-7}.$$

In this case we also have

$$\tan\gamma_c = \frac{(mR)^2}{4} \quad \text{and} \quad H_c = \frac{H_0}{1 + \frac{m^4R^4}{64}}$$

approximately.

Hence, when mR is small, H_c is practically equal to H_0 and γ_c is very nearly zero. The magnetisation therefore will be practically uniform over the cross section of the core and we may write

$$B_{\text{max.}} = \mu H_0.$$

Thus the formula may be written in the form

$$W'' = 0{\cdot}785 \frac{S}{\sigma} f^2 B^2_{\text{max.}} \cdot 10^{-7},$$

where W'' is the average loss due to eddy currents, in watts per cubic centimetre of the core, S is the area of the cross section of the core in square centimetres, σ is the resistivity of the core in C.G.S. units, f is the frequency and $B_{\text{max.}}$ is the maximum value of the flux density in C.G.S. units.

The above formula shows us that for a thin wire the higher the frequency and the greater the induction density the greater will be the core loss due to eddy currents. If we double both the induction density and the frequency, the eddy current loss will be increased sixteen times. But by using sixteen cores each having only one quarter the diameter of the original one or by using a core of the same size but of sixteen-fold resistivity the loss in the second case would be the same as in the first.

We saw earlier in the chapter that the eddy current loss, per cubic centimetre, in thin sheets was given by the formula

$$W'' = 1{\cdot}64 \frac{(2a)^2}{\sigma} f^2 B^2_{\text{max.}} \cdot 10^{-7},$$

where $2a$ is the thickness of the sheets. If we use wire of diameter $2a$, the eddy current loss per cubic centimetre would be given by

$$W'' = 0{\cdot}62 \frac{(2a)^2}{\sigma} f^2 B^2_{\text{max.}} \cdot 10^{-7},$$

and thus the eddy current loss per cubic centimetre would only be about two-fifths as great as it was in the case of the sheet.

In making calculations in connection with choking coils and transformers, where the core consists of a bundle of insulated iron wires, it is sometimes assumed that to a first approximation the core loss is given by an equation of the form

$$W_0 = V \left\{ \eta f B^{1\cdot 6}_{\text{max.}} + \theta \frac{S}{\sigma} f^2 B^2_{\text{max.}} \right\} 10^{-7},$$

where η is Steinmetz's coefficient, θ a constant and V the volume of the iron in the core in cubic centimetres. If the magnetic induction does not approximately follow the harmonic law then the term in the above formula for the eddy current losses is certainly wrong. For example, if the flux density B be practically

Tables of θ, sin θ, cos θ, sinh θ *and* cosh θ.

θ in radians	sin θ	cos θ	sinh θ	cosh θ	θ in degrees
0·00	0·000	1·000	0·000	1·000	0·000
0·01	0·010	1·000	0·010	1·000	0·573
0·02	0·020	1·000	0·020	1·000	1·146
0·03	0·030	1·000	0·030	1·000	1·719
0·04	0·040	0·999	0·040	1·001	2·292
0·05	0·050	0·999	0·050	1·001	2·865
0·06	0·060	0·998	0·060	1·002	3·438
0·07	0·070	0·998	0·070	1·002	4·011
0·08	0·080	0·997	0·080	1·003	4·584
0·09	0·090	0·996	0·090	1·004	5·157
0·10	0·100	0·995	0·100	1·005	5·730
0·20	0·199	0·980	0·201	1·020	11·46
0·30	0·295	0·955	0·305	1·045	17·19
0·40	0·389	0·921	0·411	1·081	22·92
0·50	0·479	0·878	0·521	1·128	28·65
0·60	0·565	0·825	0·637	1·185	34·38
0·70	0·644	0·765	0·759	1·255	40·11
0·80	0·717	0·697	0·888	1·337	45·84
0·90	0·783	0·622	1·027	1·433	51·57
1·00	0·842	0·540	1·175	1·543	57·30
1·10	0·891	0·453	1·336	1·669	63·03
1·20	0·932	0·362	1·509	1·811	68·75
1·30	0·964	0·268	1·698	1·971	74·48
1·40	0·985	0·170	1·904	2·151	80·21
1·50	0·997	0·071	2·129	2·352	85·94
1·60	1·000	−0·029	2·376	2·577	91·67
1·70	0·992	−0·129	2·646	2·828	97·40
1·80	0·974	−0·227	2·942	3·107	103·1
1·90	0·946	−0·324	3·268	3·418	108·9
2·00	0·909	−0·416	3·627	3·762	114·6
3·00	0·141	−0·990	10·02	10·07	171·9
4·00	−0·757	−0·653	27·29	27·31	229·2

uniform at any instant in the metal, then we can see that the eddy current loss must vary as $\left(\frac{dB}{dt}\right)^2$ and the mean value of this expression for the complete cycle depends not only on $B_{\text{max.}}$ but also on the law of variation of B with regard to the time. Owing to hysteresis, even when the magnetic induction does follow the harmonic law, the flux density does not follow it and this must modify the second term considerably. Thus when μ is not constant, the formulae given for the eddy currents induced in masses of metal must be regarded only as rough approximations.

The tables given on the opposite page and below will be of use when calculating eddy current losses.

Since $$\epsilon^n = \cosh n + \sinh n,$$

and $$\epsilon^{-n} = \cosh n - \sinh n,$$

the values of ϵ^n when n is $\pm 0{\cdot}01$, $\pm 0{\cdot}02$, ... can be calculated easily from the above table. When n is greater than 4 we may, without appreciable error, write $\sinh n = \cosh n = \frac{1}{2}\epsilon^n$.

The values of ϵ^n when n has the following values are often required.

n	ϵ^n	ϵ^{-n}	n	ϵ^n	ϵ^{-n}
$\frac{1}{10}$	1·105	0·905	$\frac{\pi}{4}$	2·193	0·456
$\frac{1}{9}$	1·117	0·895	$\frac{\pi}{2}$	4·811	0·208
$\frac{1}{8}$	1·133	0·883	$\frac{3\pi}{4}$	10·56	0·095
$\frac{1}{7}$	1·154	0·867	π	23·14	0·043
$\frac{1}{6}$	1·181	0·847	$\frac{5\pi}{4}$	50·75	0·020
$\frac{1}{5}$	1·221	0·819	$\frac{3\pi}{2}$	111·3	0·009
$\frac{1}{4}$	1·284	0·779	$\frac{7\pi}{4}$	244·2	0·004
$\frac{1}{3}$	1·396	0·717	2π	535·5	0·002
$\frac{1}{2}$	1·649	0·607	3π	12390	0·000
1	2·718	0·368	4π	286700	0·000
1·5	4·482	0·223			
2	7·389	0·135			
2·5	12·18	0·082			
3	20·09	0·050			

Tables of ber (x), bei (x), ber$'$ (x) *and* bei$'$ (x) *calculated by Magnus Maclean.*

$$\text{ber}(x) = 1 - \frac{x^4}{2^2 . 4^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \ldots\ldots,$$

$$\text{bei}(x) = \frac{x^2}{2^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \frac{x^{10}}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} - \ldots\ldots,$$

$$\text{ber}'(x) = \frac{d}{dx}\text{ber}(x) \quad \text{and} \quad \text{bei}'(x) = \frac{d}{dx}\text{bei}(x).$$

x	ber (x)	bei (x)	ber$'$ (x)	bei$'$ (x)
0·0	1·0000	0·0000	0·0000	0·0000
0·5	0·9990	0·0625	−0·0078	0·2499
1·0	0·9844	0·2496	−0·0624	0·4999
1·5	0·9211	0·5576	−0·2100	0·7303
2·0	0·7517	0·9723	−0·4931	0·9170
2·5	0·3999	1·457	−0·9436	0·9983
3·0	−0·2214	1·938	−1·570	0·8805
3·5	−1·194	2·283	−2·336	0·4353
4·0	−2·563	2·293	−3·135	−0·4911
4·5	−4·299	1·686	−3·754	−2·053
5·0	−6·230	0·116	−3·844	−4·354
5·5	−7·974	−2·790	−2·907	−7·373
6·0	−8·858	−7·335	−0·2931	−10·846
8·0	20·97	−35·02	38·29	−7·662
10·0	138·8	56·37	51·37	135·2
15·0	−2970	−2952	86·65	−4089
20·0	47580	11500	24330	41490

In applying the formulae given in this chapter the following table will be found useful.

x	$\text{ber}^2(x) + \text{bei}^2(x)$	$\{\text{ber}^2(x) + \text{bei}^2(x)\}^{-1}$	$\{\text{ber}^2(x) + \text{bei}^2(x)\}^{-\frac{1}{2}}$
0·0	1·000	1·000	1·000
0·5	1·002	0·998	0·999
1·0	1·031	0·970	0·985
1·5	1·159	0·863	0·929
2·0	1·510	0·662	0·814
2·5	2·283	0·438	0·662
3·0	3·805	0·263	0·513
3·5	6·638	0·151	0·388
4·0	$1·183 \times 10$	$0·845 \times 10^{-1}$	0·291
4·5	$2·132 \times 10$	$0·469 \times 10^{-1}$	0·217
5·0	$3·883 \times 10$	$0·258 \times 10^{-1}$	0·161
5·5	$7·137 \times 10$	$0·140 \times 10^{-1}$	0·118
6·0	$1·323 \times 10^2$	$0·756 \times 10^{-2}$	$0·869 \times 10^{-1}$
8·0	$1·666 \times 10^3$	$0·600 \times 10^{-3}$	$0·245 \times 10^{-1}$
10·0	$2·244 \times 10^4$	$0·446 \times 10^{-4}$	$0·668 \times 10^{-2}$
15·0	$1·754 \times 10^7$	$0·570 \times 10^{-7}$	$0·239 \times 10^{-3}$
20·0	$2·396 \times 10^9$	$0·417 \times 10^{-9}$	$0·204 \times 10^{-4}$

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CHAPTER XVII.

The method of duality. Ohm's law. Series and parallel. Capacity and inductance. Flux and quantity. Inductive coil and leaky condenser. Electrostatic and electromagnetic coefficients. The electromagnetic and the electrostatic wattmeter. Star and Mesh. Three phase examples. Two phase examples. References.

The method of duality.

IN geometry the method of reciprocal polars enables us, when we are given a theorem concerning points and lines, to deduce a reciprocal theorem concerning lines and points. There are many reciprocal relations of this nature in geometry, and the method of reciprocating a theorem may be called the method of duality. In electrical theory there are also many reciprocal relations, and we shall show that the method of duality often leads to important results. When the solution of the reciprocal of a problem is known, the solution of the problem can always be written down at once. The only difference in the mathematical working in the two cases is that the constants in the one equation are the reciprocals of the constants in the other. The importance of the method, however, is not so much in the saving of mathematical labour effected, as in the suggestion of novel theorems which sometimes indicate more convenient methods of making measurements or even suggest novel instruments or machines of value in electro-technics. Instead of writing down the differential equations and giving a list of the reciprocal relations at once, it will be more instructive to gradually build up these reciprocal relations and illustrate them by simple examples as we proceed. We shall first consider direct current theory.

Ohm's law may be expressed algebraically by

Ohm's Law.

$$E = CR \qquad \text{.............................(1)},$$

or

$$C = E\frac{1}{R} \qquad \text{.............................(2)}.$$

If in (1) we write C for E, E for C and $1/R$ for R we get (2). We may therefore regard (1) and (2) as reciprocal equations.

In general, if we are given any relation between E, C and R which holds for a particular theorem, we can write down at once a reciprocal relation, in which C, E, and $1/R$ are connected in the same way, which holds for the reciprocal theorem.

Series and parallel.

The equation which gives us the sum of the potential differences across the terminals of n coils in series is

$$E = C(R_1 + R_2 + \dots + R_n).$$

Reciprocating this equation, we get

$$C = E\left(\frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}\right),$$

which is the formula for the sum of the currents in n coils in parallel. Hence, in direct current problems, coils in series in the original theorem become coils in parallel in the reciprocal theorem.

The following elementary problems illustrate the method of reciprocating theorems. We shall denote the theorem by α and its reciprocal by β.

Kirchhoff's laws are reciprocal theorems.

α. For currents meeting at a point, we have

$$\Sigma C = 0.$$

β. For the potential differences round a circuit, we have

$$\Sigma E = 0.$$

α. The currents in conductors in parallel are given by

$$C_1 = \frac{\frac{1}{r_1}}{\Sigma\frac{1}{r}}C.$$

β. The potential differences across conductors in series are given by

$$E_1 = \frac{r_1}{\Sigma r} E.$$

The reader will notice that power reciprocates into power.

α. The formulae for power are

$$W = CE, \quad W = C^2R \quad \text{or} \quad W = E^2/R.$$

β. The formulae for power are

$$W = EC, \quad W = E^2/R \quad \text{or} \quad W = C^2R.$$

α. When the current in the main is constant, the currents in the branches distribute themselves so that ΣC^2R is a minimum.

β. When the applied potential difference is constant, the potential differences across coils in series arrange themselves so that $\Sigma E^2/R$ is a minimum.

α. If a resistance x (Fig. 128 a) be placed in series with the terminals of a battery of constant electromotive force E, then the power expended in x is a maximum when x equals the internal resistance R of the battery, and the maximum power is $E^2/4R$. (Stokes's Theorem.)

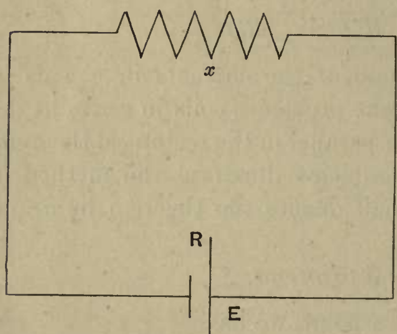

Fig. 128 a. The power expended on x is a maximum when $x = R$ (E const.).

β. If a resistance x (Fig. 128 b) be placed in parallel with a coil of resistance R and if the total current through the two coils in parallel have the constant value C, then the power expended in x is a maximum when x equals the resistance R of the coil, and the maximum power is $C^2R/4$.

The preceding theorems may be written symbolically as follows:—

(α) If $$cE = ce + c^2R \qquad (E, R \text{ constant}),$$

then ce is a maximum when c equals $E/2R$.

(β) If $$eC = ec + \frac{e^2}{R} \quad (C, 1/R \text{ constant}),$$

then ec is a maximum when e equals $CR/2$.

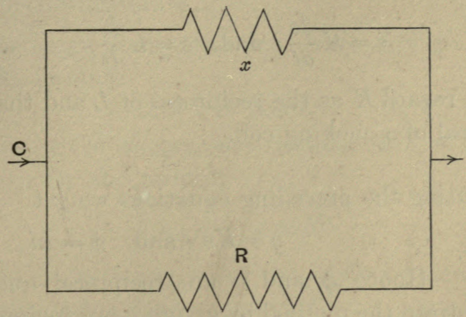

Fig. 128 *b*. The power expended on x is a maximum when $x=R$ (C const.).

α. If the potential difference between A and B (Fig. 129 *a*) be constant, the current in x is a minimum when $2x$ equals R.

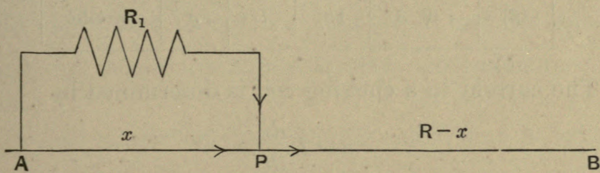

Fig. 129 *a*. The current in x is a minimum when $2x$ equals R (P.D. between A and B constant).

β. If the current in the main be constant the voltage across x is a maximum when $2x$ equals R (Fig. 129 *b*).

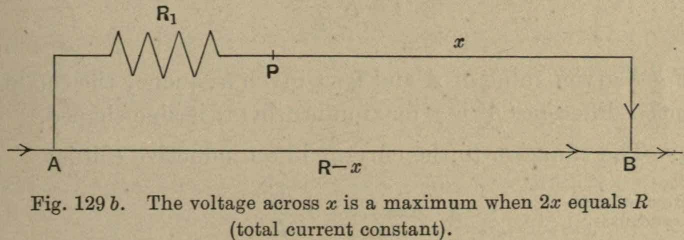

Fig. 129 *b*. The voltage across x is a maximum when $2x$ equals R (total current constant).

The method of duality is particularly suggestive when applied to alternating current theory. We remind the reader that a choking coil is an ideal coil having inductance but no resistance. The equations for a condenser and a choking coil are

Capacity and inductance.

$$i = K\frac{de}{dt} \quad \text{and} \quad e = L\frac{di}{dt}.$$

We may thus regard K as the reciprocal of L and that a condenser is the reciprocal of a choking coil.

By integrating the preceding equations we get

Flux and quantity.

$$q = Ke \quad \text{and} \quad \phi = Li.$$

Hence ϕ and q are reciprocal quantities. We shall now illustrate the method by giving a few reciprocal theorems in which use is made of the following table of reciprocal quantities and connections.

(α)	e or V	r	K	ϕ	series
(β)	i or A	$1/r$	L	q	parallel

α. The current in a choking coil is determined by

$$e = L\frac{di}{dt}.$$

For a given value of the effective voltage V, and for a given frequency, the effective value of the choking coil current A is a maximum when e is sine shaped.

β. The voltage across a condenser is determined by

$$i = K\frac{de}{dt}.$$

For a given value of A and for a given frequency the condenser potential difference V is a maximum when i is sine shaped.

α. The equation to the current in an inductive coil is

Inductive coil and leaky condenser.

$$e = Ri + L\frac{di}{dt}.$$

β. This reciprocates into

$$i = \frac{e}{R} + K\frac{de}{dt}.$$

An inductive coil therefore reciprocates into a condenser shunted by a non-inductive resistance.

α. In sine curve theory, the impedance Z of an inductive coil is given by the formula

$$Z = \sqrt{R^2 + \omega^2 L^2}.$$

β. In sine curve theory, the impedance Z of a leaky condenser is given by the formula

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \omega^2 K^2}.$$

α. In sine curve theory, when the applied potential difference is constant, the mean power expended at a given frequency, in a variable resistance in series with a choking coil, is a maximum when R is $L\omega$. (J. Hopkinson.)

β. In sine curve theory, when the main current is constant the mean power expended at a given frequency, in a variable resistance shunting a condenser, is a maximum when $1/R$ is $K\omega$.

α. In sine curve theory, a condenser K may be replaced by a choking coil whose resistance is zero and inductance $-\dfrac{1}{K\omega^2}$. (Rayleigh.)

β. In sine curve theory, a choking coil L may be replaced by a condenser whose resistance is infinite and capacity $-\dfrac{1}{L\omega^2}$.

α. In sine curve theory, when the current in a resistance coil and choking coil in series lags behind the applied potential difference by an angle θ, then

$$\sin\theta = \frac{\omega L}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}}; \quad \cos\theta = \frac{R}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}}; \quad \tan\theta = \frac{\omega L}{R}.$$

β. In sine curve theory, when the potential difference between the terminals of a resistance coil and a condenser in parallel lags behind the main current by an angle θ, then

$$\sin\theta = \frac{\omega K}{\left(\frac{1}{R^2} + \omega^2 K^2\right)^{\frac{1}{2}}}; \quad \cos\theta = \frac{\frac{1}{R}}{\left(\frac{1}{R^2} + \omega^2 K^2\right)^{\frac{1}{2}}}; \quad \tan\theta = \omega K R.$$

α. In sine curve theory, if θ be the angle of phase difference between the current and the potential difference applied to a resistance coil, a choking coil and a condenser in series, then

$$\tan\theta = \frac{\omega\left(L - \frac{1}{K\omega^2}\right)}{R}.$$

β. In sine curve theory, if θ be the angle of phase difference between the potential difference and the current supplying a resistance coil, a condenser and a choking coil in parallel, then

$$\tan\theta = \omega R\left(K - \frac{1}{L\omega^2}\right).$$

α. The power expended in n leaky condensers in series is

$$\frac{V_1^2}{R_1} + \frac{V_2^2}{R_2} + \ldots\ldots.$$

β. The power expended in n coils in parallel is

$$A_1^2 R_1 + A_2^2 R_2 + \ldots\ldots$$

whether the coils are inductive or not.

α. The power factor of an inductive coil is given by

$$\cos\phi = \frac{RA}{V}.$$

β. The power factor of a leaky condenser is given by

$$\cos\phi = \frac{V}{RA}.$$

α. The resonance of electromotive forces. When a condenser and a choking coil are in series and the effective value of the applied potential difference is constant, the effective potential difference across the terminals of either attains maximum values when

$$LK\{(2n+1)\,\omega\}^2 = 1.$$

β. The resonance of currents. When a choking coil and a condenser are in parallel and the main current is constant, the currents in either of them attain maximum values when

$$KL\,\{(2n+1)\,\omega\}^2 = 1.$$

α. In sine curve theory, when the effective value of the main current is constant, the effective current in an inductive coil (Fig. 130 a) shunted by a condenser is a maximum, when

$$K = \frac{L}{R^2 + \omega^2 L^2}.$$

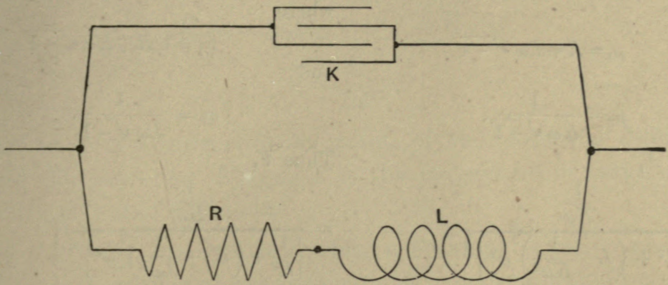

Fig. 130 a. The current in the coil L is a maximum when $K = \frac{L}{R^2 + \omega^2 L^2}$. The main current is constant.

β. In sine curve theory, when the applied potential difference is constant, the potential difference across a leaky condenser, which is put in series with a choking coil, is a maximum when the self inductance L is given by

$$L = \frac{K}{1/R^2 + \omega^2 K^2}.$$

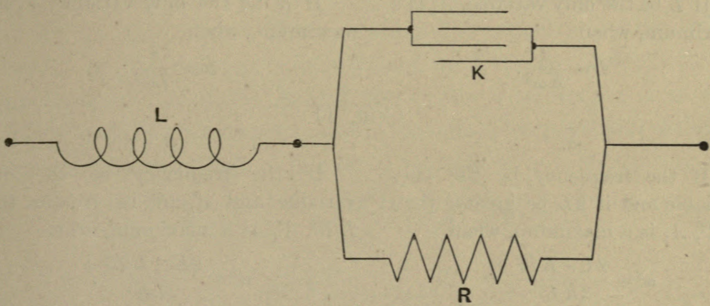

Fig. 130 b. The P.D. across K is a maximum when it is put in series with a coil whose self inductance $L = \frac{K}{\frac{1}{R^2} + \omega^2 K^2}$. The applied P.D. is constant.

The above theorems, which are of importance in practical work, can easily be proved graphically. The main steps in the proof, by the method of Chap. VII, may be written as follows:—

α.

Inductive coil shunted by a condenser.

$$[i_1] = \frac{[\rho_2]}{[\rho_1+\rho_2]}[i],$$

where

$$\rho_1 = R + L\omega\sqrt{-1}$$

and

$$\rho_2 = \frac{1}{K\omega\sqrt{-1}}.$$

Thus A_1

$$= \frac{\frac{1}{K\omega}}{\left\{R^2+\left(L-\frac{1}{K\omega^2}\right)^2\omega^2\right\}^{\frac{1}{2}}}A$$

$$= \frac{A}{\{K^2\omega^2(R^2+L^2\omega^2)-2LK\omega^2+1\}^{\frac{1}{2}}}.$$

Hence, if K be the only variable, A_1 is a maximum, when

$$K = \frac{L}{R^2+L^2\omega^2}.$$

β.

Leaky condenser in series with a choking coil.

$$[e_1] = \frac{[\sigma_2]}{[\sigma_1+\sigma_2]}[e],$$

where

$$\sigma_1 = 1/R + K\omega\sqrt{-1}$$

and

$$\sigma_2 = \frac{1}{L\omega\sqrt{-1}}.$$

Thus V_1

$$= \frac{\frac{1}{L\omega}}{\left\{\frac{1}{R^2}+\left(K-\frac{1}{L\omega^2}\right)^2\omega^2\right\}^{\frac{1}{2}}}V$$

$$= \frac{V}{\{L^2\omega^2(1/R^2+K^2\omega^2)-2KL\omega^2+1\}^{\frac{1}{2}}}.$$

Hence, if L be the only variable, V_1 is a maximum, when

$$L = \frac{K}{1/R^2+K^2\omega^2}.$$

COR. I.

α.

If L be the only variable, A_1 is a maximum, when

$$L = \frac{1}{K\omega^2}.$$

β.

If K be the only variable, V_1 is a maximum, when

$$K = \frac{1}{L\omega^2}.$$

COR. II.

α.

If the frequency be the only variable and if $2L$ be greater than KR^2, A_1 is a maximum, when

$$\omega^2 = \frac{2L-KR^2}{2KL^2}.$$

β.

If the frequency be the only variable and if $2K$ be greater than L/R^2, V_1 is a maximum, when

$$\omega^2 = \frac{2K-L/R^2}{2LK^2}.$$

α. If we have n coils in parallel and if their time constants are all equal and their mutual inductances zero, so that

$$\frac{L_1}{R_1} = \frac{L_2}{R_2} = \ldots\ldots = \frac{L_n}{R_n},$$

then

$$\dot{i}_1 = \frac{1/R_1}{\Sigma 1/R}\dot{i},$$

where i_1 is the instantaneous value of the current in the coil (R_1, L_1) and i is the instantaneous value of the current in the main.

β. If we have n leaky condensers in series and if their time constants are all equal, so that

$$K_1R_1 = K_2R_2 = \ldots\ldots = K_nR_n,$$

then

$$v_1 = \frac{R_1}{\Sigma R} v,$$

where v_1 is the instantaneous value of the potential difference across the condenser K_1, and v is the instantaneous value of the applied potential difference.

α. In sine curve theory, when a constant effective potential difference is applied at the terminals of a non-inductive resistance R in series with a shunted choking coil, the effective current in the main is a minimum when

$$x = \frac{\omega^2L^2}{2R} + \frac{\omega L}{2R}\{\omega^2L^2 + 4R^2\}^{\frac{1}{2}},$$

or

$$\frac{1}{x} = \left\{\frac{1}{4R^2} + \frac{1}{\omega^2L^2}\right\}^{\frac{1}{2}} - \frac{1}{2R},$$

where x is the non-inductive resistance shunting the choking coil L. This is a particular case of a theorem given on page 168.

β. In sine curve theory, when the effective value of the current in the main is constant and we have two branch circuits, one being a resistance R and the other a condenser K in series with a variable resistance x, then the effective potential difference between the terminals of the fixed resistance is a maximum, when

$$\frac{1}{x} = \frac{\omega^2K^2R}{2} + \frac{\omega KR}{2}\{\omega^2K^2 + 4/R^2\}^{\frac{1}{2}},$$

or

$$x = \left\{\frac{R^2}{4} + \frac{1}{\omega^2K^2}\right\}^{\frac{1}{2}} - \frac{R}{2}.$$

The general theorem given on page 168 may be reciprocated in the same manner (see Figs. 131 *a* and 131 *b*).

α. In sine curve theory, if we have a condenser K in series with a resistance R, and if the combination be shunted by a choking coil L in series with a resistance R, then if $LK\omega^2 = 1$, the combination is equivalent to a non-inductive coil whose resistance R' is given by

$$R' = \frac{L}{2KR} + \frac{R}{2}.$$

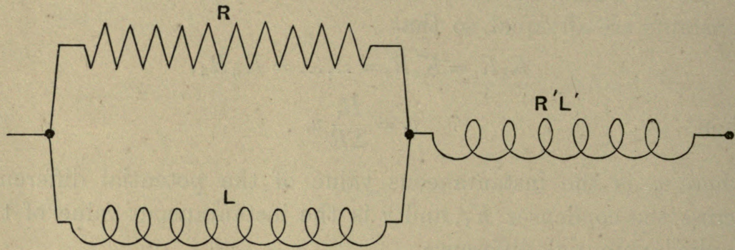

Fig. 131 *a*. The main current is a minimum when R has a certain value. The applied P.D. is constant.

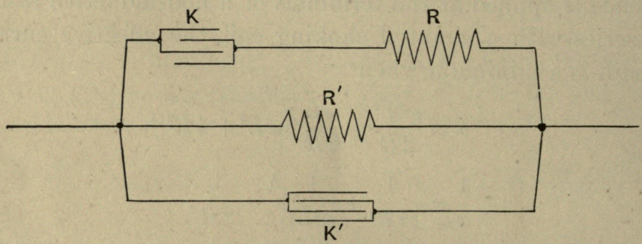

Fig. 131 *b*. The P.D. is a maximum when R has a certain value. The main current is constant.

β. In sine curve theory, if we have a choking coil L shunted by a resistance R, and if the combination be in series with a condenser K shunted by a resistance R, then if $KL\omega^2 = 1$, the combination is equivalent to a non-inductive coil whose resistance R' is given by

$$\frac{1}{R'} = \frac{KR}{2L} + \frac{1}{2R}.$$

The following is an outline of the analytical proof of the above theorems arranged so as to show that from the mathematical point of view the two problems are identical.

α.	β.
$e = Ri_1 + L\dfrac{di_1}{dt}$	$i = \dfrac{e_1}{R} + K\dfrac{de_1}{dt}$
$e = Ri_2 + \dfrac{\int i_2 dt}{K}$.	$i = \dfrac{e_2}{R} + \dfrac{\int e_2 dt}{L}$.

If the functions follow the harmonic law and if we differentiate twice the equations containing the integral sign and divide by $-\omega^2$, we obtain

$$e = Ri_2 - \frac{1}{K\omega^2}\frac{di_2}{dt}. \qquad i = \frac{e_2}{R} - \frac{1}{L\omega^2}\frac{de_2}{dt}.$$

Assuming

$$e = E \sin \omega t,$$

we get $$i_1 = \frac{E \sin(\omega t - \alpha)}{\sqrt{R^2 + L^2\omega^2}},$$

and $$i_2 = \frac{E \sin(\omega t + \alpha)}{\sqrt{R^2 + L^2\omega^2}},$$

where

$$\tan \alpha = L\omega/R.$$

Hence

$$\frac{E \sin \omega t}{i_1 + i_2} = \frac{\sqrt{R^2 + L^2\omega^2}}{2 \cos \alpha},$$

and $$R' = \frac{R^2 + L^2\omega^2}{2R} = \frac{R}{2} + \frac{L}{2KR}.$$

Assuming

$$i = I \sin \omega t,$$

we get $$e_1 = \frac{I \sin(\omega t - \alpha)}{\sqrt{1/R^2 + K^2\omega^2}},$$

and $$e_2 = \frac{I \sin(\omega t + \alpha)}{\sqrt{1/R^2 + K^2\omega^2}},$$

where

$$\tan \alpha = KR\omega.$$

Hence

$$\frac{I \sin \omega t}{e_1 + e_2} = \frac{\sqrt{1/R^2 + K^2\omega^2}}{2 \cos \alpha},$$

and $$1/R' = \frac{1/R^2 + K^2\omega^2}{2/R} = \frac{1}{2R} + \frac{K}{2LR}.$$

α. When a condenser K (Fig. 132 a) shunted by a resistance R is placed in series with a choking coil L shunted by a resistance R, the combination will act like a non-inductive resistance R at all frequencies and whatever the shape of the wave of the applied potential difference, provided that L equals KR^2 (see page 87).

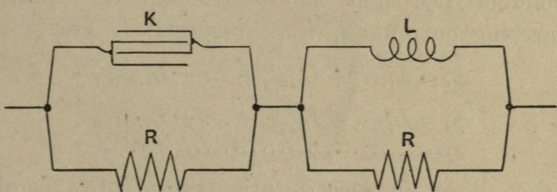

Fig. 132 a. When L equals KR^2 the combination acts like a non-inductive resistance R.

β. When a choking coil L (Fig. 132 b) in series with a resistance R is placed in parallel with a condenser K in series with a resistance R, the combination will act like a non-inductive resistance R at all frequencies and whatever the shape of the wave of the main current, provided that K equals L/R^2 (see page 86).

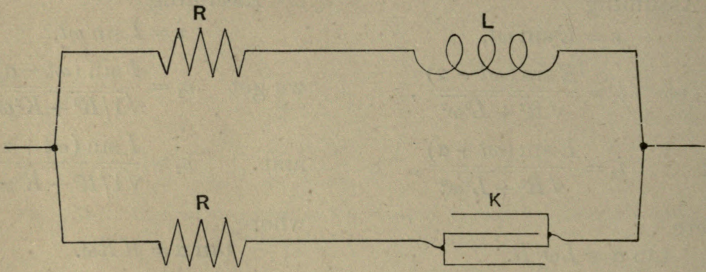

Fig. 132 b. When K equals L/R^2 the combination acts like a non-inductive resistance R.

Electrostatic and electromagnetic coefficients.

Let us now consider whether the coefficients of self and mutual induction for electrostatic charges have the corresponding electromagnetic coefficients for their reciprocals. Maxwell's equations for the electrostatic charges in terms of the potentials of n conductors (page 90) are

$$q_1 = K_{1.1}v_1 + K_{1.2}v_2 + \dots + K_{1.n}v_n,$$

$$q_2 = K_{2.1}v_1 + K_{2.2}v_2 + \dots + K_{2.n}v_n,$$

$$\dots\dots\dots\dots\dots\dots$$

If ϕ_p be the flux through a circuit p, which has $L_{p.p}$ and $L_{p.q}$ for its self and mutual inductances respectively, and the current i_p entirely surrounds the flux ϕ_p, as it does when the circuit has infinite conductivity, then the electromagnetic equations to n circuits carrying currents i_1, i_2, ... respectively are

$$\phi_1 = L_{1.1}i_1 + L_{1.2}i_2 + \dots + L_{1.n}i_n,$$

$$\phi_2 = L_{2.1}i_1 + L_{2.2}i_2 + \dots + L_{2.n}i_n,$$

$$\dots\dots\dots\dots\dots\dots$$

We see, then, that an electrostatic system of n conductors at given potentials reciprocates into an electromagnetic system of n circuits carrying given currents, if we assume that $L_{p.p}$ is the reciprocal of $K_{p.p}$ and that $L_{p.q}$ is the reciprocal of $K_{p.q}$. In the above equations it is to be noted that

$$K_{p.q} = K_{q.p} \quad \text{and} \quad L_{p.q} = L_{q.p}.$$

It is also to be noticed that when p and q are different $K_{p.q}$ is always negative but $L_{p.q}$ is not necessarily negative. We have supposed that the conductors and circuits or coils have perfect conductivity.

By means of determinants, Maxwell's equations can be written in the form

$$v_1 = p_{1.1}q_1 + p_{1.2}q_2 + \dots + p_{1.n}q_n,$$
$$v_2 = p_{2.1}q_1 + p_{2.2}q_2 + \dots + p_{2.n}q_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

where $p_{l.m}$ is given by the equation

$$p_{l.m}\Delta = M_{l.m},$$

Δ being the symmetrical determinant $\Sigma \pm K_{1.1}K_{2.2} \dots K_{n.n}$, and $M_{l.m}$ being the coefficient of $K_{l.m}$ in Δ. Maxwell calls $p_{1.1}, p_{1.2}, \dots$ coefficients of potential.

Similarly in the electromagnetic problem we have

$$i_1 = \lambda_{1.1}\phi_1 + \lambda_{1.2}\phi_2 + \dots + \lambda_{1.n}\phi_n$$

and $(n-1)$ similar equations, where

$$\lambda_{l.m}\Delta' = M'_{l.m},$$

Δ' being the symmetrical determinant $\Sigma \pm L_{1.1}L_{2.2} \dots L_{n.n}$ and $M'_{l.m}$ being the coefficient of $L_{l.m}$ in Δ'. We may call $\lambda_{1.1}, \lambda_{1.2}, \dots$ coefficients of current. The above equations prove that $\lambda_{l.m}$ is the reciprocal of $p_{l.m}$.

a. The capacity K_v for equal potentials of n conductors is given by

$$K_v = \Sigma K_{1.1} + 2\Sigma K_{1.2},$$

for when they are all at the same potential v,

$$q_1 = (K_{1.1} + K_{1.2} + \dots + K_{1.n})\, v,$$
$$q_2 = (K_{2.1} + K_{2.2} + \dots + K_{2.n})\, v,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

and hence $$K_v = (\Sigma q)/v = \Sigma K_{1.1} + 2\Sigma K_{1.2},$$

since $$K_{p.q} = K_{q.p}.$$

The electrostatic energy of the system in this case is $\frac{1}{2}v\Sigma q$ or $\frac{1}{2}K_v v^2$. It follows from a theorem proved in a note at the end of this chapter that, if v be maintained constant, the mutual electrical actions of the conductors will tend to move them so that $\frac{1}{2}K_v v^2$ increases. Thus as the conductors separate under the action of the electric forces, K_v continually increases.

β. The self inductance L_s of n coils in series is given by

$$L_s = \Sigma L_{1.1} + 2\Sigma L_{1.2},$$

for when they are all carrying the same current i,

$$\phi_1 = (L_{1.1} + L_{1.2} + \ldots + L_{1.n})\, i,$$

$$\phi_2 = (L_{2.1} + L_{2.2} + \ldots + L_{2.n})\, i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and hence $$L_s = (\Sigma\phi)/i = \Sigma L_{1.1} + 2\Sigma L_{1.2},$$

since $$L_{p.q} = L_{q.p}.$$

The electromagnetic energy in this case is $\frac{1}{2}i\Sigma\phi$ or $\frac{1}{2}L_s i^2$. If i be maintained constant, the mutual electrical actions of the various coils will tend to move them so that $\frac{1}{2}L_s i^2$ increases. Thus, as the coils separate under the action of the electromagnetic forces, L_s continually increases.

If we have two coils in parallel carrying currents i_1 and i_2 and if $\frac{d\phi}{dt}$ be the applied potential difference and the total current in each of the neighbouring circuits is zero, then

$$\phi = L_{1.1}i_1 + L_{1.2}i_2,$$

and $$\phi = L_{2.1}i_1 + L_{2.2}i_2.$$

Hence $$\phi\,(L_{2.2} - L_{1.2}) = (L_{1.1}L_{2.2} - L^2_{1.2})\, i_1,$$

and $$\phi\,(L_{1.1} - L_{1.2}) = (L_{1.1}L_{2.2} - L^2_{1.2})\, i_2.$$

Therefore $$\frac{\phi}{i_1 + i_2} = \frac{L_{1.1}L_{2.2} - L^2_{1.2}}{L_{1.1} + L_{2.2} - 2L_{1.2}}.$$

But $i_1 + i_2$ is the current in the main and therefore $\phi/(i_1 + i_2)$ is the self inductance of the coils in parallel. Now leaving the positions of the coils unchanged, reverse the connections of the coil 2, so that the applied potential difference is acting on the coil 1 in the same direction as in the last example but on the coil 2 in the opposite direction. The equations become

$$\phi = L_{1.1} i_1 + L_{1.2} i_2,$$

$$-\phi = L_{2.1} i_1 + L_{2.2} i_2,$$

and therefore
$$\frac{\phi}{i_1 - i_2} = \frac{L_{1.1} L_{2.2} - L^2_{1.2}}{L_{1.1} + L_{2.2} + 2L_{1.2}}.$$

Now, in this case, $i_1 - i_2$ is the current in the main. Hence, when the two coils are connected in 'cross parallel,' the self inductance is

$$\frac{L_{1.1} L_{2.2} - L^2_{1.2}}{L_{1.1} + L_{2.2} + 2L_{1.2}}.$$

If we reciprocate this expression we get

$$\frac{K_{1.1} K_{2.2} - K^2_{1.2}}{K_{1.1} + K_{2.2} + 2K_{1.2}},$$

and this equals $q/(v_1 - v_2)$, that is, according to our definition in Chapter IV, the capacity between the two conductors when all the others are earthed.

We see, therefore, that the self inductance of two coils in 'cross parallel,' when the total current in each of the neighbouring coils is zero, reciprocates into the capacity between two conductors when all neighbouring conductors are earthed.

If we reciprocate the formula for the inductance of two coils in parallel we get

$$\frac{q}{v_1 + v_2} = \frac{K_{1.1} K_{2.2} - K^2_{1.2}}{K_{1.1} + K_{2.2} - 2K_{1.2}}.$$

This formula is derived from the equations

$$q = K_{1.1} v_1 + K_{1.2} v_2,$$

and
$$q = K_{2.1} v_1 + K_{2.2} v_2.$$

If two conductors, therefore, have equal charges and all the other conductors in the neighbourhood are at zero potential, the ratio of the charge on either conductor to the sum of the potentials of the two is a constant. In practice it is not convenient to give equal charges of the same sign to conductors and thus this constant ratio is, at present, mainly of theoretical importance.

α. The self inductance of n circuits in parallel is given by

$$1/L_p = (\Sigma i)/\phi = \Sigma\lambda_{1.1} + 2\Sigma\lambda_{1.2},$$

where $\lambda_{1.1}, \lambda_{1.2}, \ldots$ are the current coefficients. The electromagnetic energy of the circuits is $\phi^2/2L_p$.

β. The capacity K_q for equal charges of n conductors is given by

$$1/K_q = (\Sigma v)/q = \Sigma p_{1.1} + 2\Sigma p_{1.2},$$

where $p_{1.1}, p_{1.2}, \ldots$ are the potential coefficients. The electrostatic energy of the conductors is $q^2/2K_q$. It is easy to see that the mutual electric forces acting on the conductors tend to increase K_q.

α. When we have a system of n circuits in parallel, the ratio of ϕ to i_1 is constant and is called the effective inductance of the first circuit.

β. When we have n conductors each of which has a charge q, the ratio of q to v_1 is constant and is called the effective capacity of the first conductor.

It is worth noting that, when all the charges on a system of conductors are equal and the charges on all neighbouring conductors are zero, the electrostatic energy is $q^2/2K_q$, and thus depends only on the charge, the geometrical configuration of the system and its position. Similarly, when all the potentials are maintained at a known value, the electrostatic energy can be written down at once when K_v is known. Now, in many cases, K_v can be measured easily experimentally, and in some cases K_p can be found. A knowledge of the value of these quantities will be helpful, therefore, in studying the electrical properties of a system of fixed conductors. Similarly a knowledge of L_s and L_p will be a help in studying a system of coils.

We have shown that the following quantities are reciprocals:—

$K_{p.p}$	$K_{p.q}$	The capacity between two conductors
$L_{p.p}$	$L_{p.q}$	The self-inductance of two coils in cross parallel

Denoting the coefficients of current and potential by $\lambda_{1.1}$, $\lambda_{1.2}, \ldots$ and $p_{1.1}$, $p_{1.2}, \ldots$ we also have:—

K_v	K_q	$p_{1\cdot 1}$	$p_{1\cdot 2}$
L_s	L_p	$\lambda_{1\cdot 1}$	$\lambda_{1\cdot 2}$

α. Formula for the electromagnetic wattmeter:—

$$P = ki_1i_2.$$

The electromagnetic and the electrostatic wattmeter.

One practically non-inductive coil shunts the load, and the inductive coil is in series with the load.

β. Formula for the electrostatic wattmeter:—

$$P = ke_1e_2.$$

One resistance which has practically no condenser effect is placed in series with the load, and a condenser (formed by the needle and quadrants) is placed shunting the load.

α. The three voltmeter method of measuring power:—

$$P = \frac{1}{2R}(V^2 - V_1^2 - V_2^2).$$

β. The three ammeter method of measuring power:—

$$P = \frac{R}{2}(A^2 - A_1^2 - A_2^2).$$

These examples could easily be multiplied.

Star and mesh.

An inspection of Chapters XI and XII will show that many of the theorems in two and three phase theory are reciprocals. In a star connection Σi is zero and in a mesh connection Σv is zero. Hence theorems concerning currents in star systems reciprocate into theorems concerning potential differences in mesh systems. The following are a few illustrations.

Three phase examples.

α. The mesh potential differences between the mains are to one another as the sines of the phase differences between them. In symbols, we have

$$\frac{V_{2.3}}{\sin\alpha} = \frac{V_{3.1}}{\sin\beta} = \frac{V_{1.2}}{\sin\gamma}.$$

β. The currents in the arms of a star load are to one another as the sines of the phase differences between them. In symbols, we have

$$\frac{A_1}{\sin\alpha}=\frac{A_2}{\sin\beta}=\frac{A_3}{\sin\gamma}.$$

α. In a non-inductive star load, if p, q and r be the resistances of the arms,

$$\frac{E_1}{p\sin\theta_{2.3}}=\frac{E_2}{q\sin\theta_{3.1}}=\frac{E_3}{r\sin\theta_{1.2}}.$$

β. In a non-inductive mesh load, if p, q and r be the resistances of the arms,

$$\frac{I_1 p}{\sin\theta_{2.3}}=\frac{I_2 q}{\sin\theta_{3.1}}=\frac{I_3 r}{\sin\theta_{1.2}}.$$

α. If the load consist of three equal non-inductive resistances connected mesh fashion, then

$$A_1^4+A_2^4+A_3^4=9\,(I_1^4+I_2^4+I_3^4),$$

where A_1, A_2 and A_3 are the currents in the three mains and I_1, I_2 and I_3 are the mesh currents.

β. If the load consist of three equal non-inductive resistances connected star fashion, then

$$V^4_{2.3}+V^4_{3.1}+V^4_{1.2}=9\,(E_1^4+E_2^4+E_3^4).$$

α. The formula for the potential differences across one of the arms of a non-inductive star load is

$$E_1^2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\left(\frac{V^2_{1.2}}{r_2}+\frac{V^2_{3.1}}{r_3}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)-\frac{V^2_{2.3}}{r_2 r_3}.$$

β. The formula for the current in one side of a non-inductive mesh load is

$$I_1^2\,(r_1+r_2+r_3)^2=(A_3^2 r_2+A_2^2 r_3)\,(r_2+r_3)-A_1^2 r_2 r_3.$$

α. The mesh voltages being constant, the sum of the voltages between the centre of a star load and the three mains is a minimum when the currents in the arms are all equal.

β. The currents in the mains being constant the sum of the mesh currents is a minimum when the potential differences between the mains are equal.

Two phase examples.

An inspection of the formulae and constructions, given in Chapter XII for two phase theory, will show how easy it is to reciprocate many of them.

α. Rule for finding the voltages across the arms of a non-inductive star load when the voltage tetrahedron and the resistances of the arms are given.

Find the centre of gravity G of masses $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ and $\frac{1}{r_4}$ placed at the four angular points of the voltage tetrahedron; then, the lines joining G to the four angular points will give the magnitudes and the phase differences of the required voltages.

β. Rule for finding the currents in a non-inductive mesh load when the current tetrahedron and the resistances of the arms are given.

Find the centre of gravity G of masses r_1, r_2, r_3 and r_4 placed at the four angular points of the current tetrahedron; then, the lines joining G to the four angular points will give the magnitudes and the phase differences of the required currents.

(α) The power w expended on a star load (p. 256) is given by

$$w = v_1 a_1 + v_2 a_2 + v_3 a_3 + v_4 a_4.$$

(β) The power w expended on a mesh load is given by

$$w = i_{2.3} v_{2.3} + i_{3.4} v_{3.4} + i_{4.1} v_{4.1} + i_{1.2} v_{1.2}.$$

The above examples could easily be multiplied, but sufficient have been given to prove that the method of duality is as useful in electrical theory as it is in geometry.

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ADDITIONS.

NOTE ON A THEOREM IN ELECTROSTATICS.

ON p. 192 it is assumed that when the potentials of a system of conductors, one of which is movable, are maintained constant, the work done on the movable one, during an infinitely small displacement, is equal to the gain in the electrostatic energy of the system. This is a particular case of a well-known theorem in electrostatics which can be proved as follows.

Charges constant. Let us consider the case of n charged conductors which form a self-contained system screened from outside electric influences. To simplify the problem let us suppose that only one of them is movable. Let us also suppose, in the first place, that all the conductors are insulated, so that the charges on them remain constant. By the Conservation of Energy the movable conductor X cannot, unless acted on by external forces, move under the action of the electric forces into a position where the electrostatic energy of the system is greater, otherwise the total energy of the system would be increased. The electric forces acting on X move it in the direction along which the electrostatic energy diminishes most rapidly, as the force acting on X will be greatest in this direction. We see therefore that the motion which ensues, due to the action of the electric forces, must diminish the electrostatic energy of the system and increase the mechanical energy.

Let dw denote the work done on X by the electric forces during an infinitely small displacement from P to P', then, with the notation of p. 90, if $v_1 + dv_1$, $v_2 + dv_2$, ... be the new values of v_1, v_2, ... we have

$$\begin{aligned} dw &= \tfrac{1}{2}q_1 v_1 + \tfrac{1}{2}q_2 v_2 + \dots \\ &\quad - \tfrac{1}{2}q_1(v_1 + dv_1) - \tfrac{1}{2}q_2(v_2 + dv_2) - \dots \\ &= -\tfrac{1}{2}q_1 dv_1 - \tfrac{1}{2}q_2 dv_2 - \dots \qquad (1). \end{aligned}$$

Now q_1 is constant, and since (p. 90)

$$q_1 = K_{1.1}v_1 + K_{1.2}v_2 + \ldots,$$

we have

$$0 = K_{1.1}dv_1 + v_1 dK_{1.1} + K_{1.2}dv_2 + v_2\, dK_{1.2} + \ldots,$$

and therefore

$$K_{1.1}dv_1 + K_{1.2}dv_2 + \ldots = -v_1 dK_{1.1} - v_2 dK_{1.2} - \ldots.$$

Hence, by substituting in (1), we get

$$\begin{aligned} dw &= -\tfrac{1}{2}(K_{1.1}v_1 + K_{1.2}v_2 + \ldots)\,dv_1 - \tfrac{1}{2}(K_{2.1}v_1 + K_{2.2}v_2 + \ldots)dv_2 - \ldots \\ &= -\tfrac{1}{2}v_1(K_{1.1}dv_1 + K_{1.2}dv_2 + \ldots) - \tfrac{1}{2}v_2(K_{2.1}dv_1 + K_{2.2}dv_2 + \ldots) - \ldots \\ &= \tfrac{1}{2}v_1(v_1 dK_{1.1} + v_2 dK_{1.2} + \ldots) + \tfrac{1}{2}v_2(v_1 dK_{2.1} + v_2 dK_{2.2} + \ldots) + \ldots \\ &= \tfrac{1}{2}v_1^2 dK_{1.1} + \ldots + v_1 v_2 dK_{1.2} + \ldots. \end{aligned}$$

Potentials constant. When the potentials of the n conductors are maintained constant and have the same initial values as in the last case, the difference between the values of the electrostatic energy when X is in the positions P' and P equals

$$\tfrac{1}{2}(K_{1.1} + dK_{1.1})\,v_1^2 + \ldots + (K_{1.2} + dK_{1.2})\,v_1v_2 + \ldots$$
$$- \tfrac{1}{2}K_{1.1}v_1^2 - \ldots - K_{1.2}v_1v_2 - \ldots.$$

By the preceding paragraph this is equal to dw and is therefore positive. Hence the electrostatic energy in the position P' is greater than in the position P. Also, since the force acting on X in the position P is exactly the same in the two cases, and P' is infinitely near to P, so that, the electrostatic field being only infinitesimally disturbed by the motion of X, the force at P' is practically the same in the two cases. The mechanical work done on X, therefore, during an infinitesimal displacement is the same whether the charges or the potentials are maintained constant.

Thus the work done on X, when the potentials are constant, is also dw, and this equals the gain in the electrostatic energy of the system. It follows, by integration, that the total work done on X during a finite displacement, when the potentials are constant, equals the gain in the electrostatic energy of the system. In an electrostatic voltmeter, for instance, the energy taken from the mains equals twice the mechanical energy required to displace the moving part.

The above theorems can be stated more generally as follows:

(1) When, in a system of insulated conductors, the relative positions of the conductors alter owing to their mutual electric actions, the conductors move in such a way that the electrostatic energy is diminished, the diminution being equal to the work done on the conductors by the electric forces.

(2) When, in a system of conductors, the potentials of which are maintained constant by means of external sources, the relative positions of the conductors alter owing to their mutual electric actions, the conductors move in such a way that the electrostatic energy is increased by an amount exactly equal to the work done on the conductors by the electric forces.

Reciprocal theorems.

It is instructive to reciprocate these general theorems by the method of Chapter XVII. Reciprocating the second, we see that when the currents in a system of n coils are maintained constant, the motion due to their attractions or repulsions increases the electromagnetic energy of the system. We see also, that the mechanical work done on the coils is exactly equal to the increase of the electromagnetic energy by the electromagnetic forces.

This theorem can be proved at once by means of Lagrange's general equations of motion (see Maxwell's *Electricity and Magnetism*, Vol. II. § 580). Maxwell notes the resemblance with the corresponding electrostatic problem. The theorem was given by Sir William Thomson (Lord Kelvin) in the second edition of Nichol's *Cyclopœdia of Physical Science* (Article, "Magnetism, Dynamical Relations of") published in 1860. See also *Papers on Electrostatics and Magnetism* by Sir William Thomson, p. 446, Second Edition.

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PRESS NOTICES OF THE FIRST EDITION.

Times.—"It is difficult to think of a single branch of the physical sciences in which these advances are not of fundamental importance. The physicist sees the relations between electricity and matter laid bare in a manner hardly hoped for hitherto....The workers in the field of Science are to-day reaping an unparalleled harvest, and we may congratulate ourselves that in this field at least we more than hold our own among the nations of the world."

Engineering.—"The subject with which it deals is, perhaps, the most vitally important in the whole field covered by the name "physics" to-day... No one will doubt that for such a work the best possible author is Professor J. J. Thomson. For it is to him, more than to any other person, that we owe the rapid progress that has been made in recent years in this subject.... The work is one of the highest importance, and must be given a prominent place among the standard literature of modern science."

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